Problem 1 *(DPLL)* (10 points)

Prove the (un-)satisfiability of the following set of propositional clauses using the Davis-Putnam-Logemann-Loveland procedure. What is the minimal number of branching steps that is required by the DPLL procedure for this input?

$$
\begin{array}{rcccccccccl}
P_1 & \vee & \neg P_2 & \vee & P_3 & \vee & \neg P_4 & \vee & \neg P_5 & & (1)\\
P_1 & & & \vee & P_3 & \vee & P_4 & \vee & P_5 & & (2)\\
P_1 & & & \vee & \neg P_3 & \vee & P_4 & & & & (3)\\
P_1 & & & \vee & P_3 & \vee & \neg P_4 & \vee & P_5 & & (4)\\
P_1 & & & \vee & \neg P_3 & \vee & \neg P_4 & & & & (5)\\
\neg P_1 & \vee & \neg P_2 & & & & & & & & (6)\\
 & & P_2 & & & & & & & & (7)\\
 & & \neg P_2 & \vee & P_3 & \vee & P_4 & \vee & \neg P_5 & & (8)
\end{array}
$$

Problem 2 *(Semantics)* (10 + 10 = 20 points)

Let Σ be a signature containing at least one constant symbol, let F be a Σ-formula such that x is the only free variable in F.

Part (a)

Prove: If $\exists x\, F$ is valid, then there exists a ground Σ-term t such that $F[t/x]$ is satisfiable.

Part (b)

Refute: If $\exists x\, F$ is valid, then there exists a ground Σ-term t such that $F[t/x]$ is valid. (Hint: F may contain quantifiers and/or equations.)

Problem 3 *(Rewrite systems)* (10 points)

Is the rewrite system

$$\{\, f(a) \to f(b),\ f(b) \to f(c),\ f(c) \to f(a),\ f(x) \to x \,\}$$

(i) terminating, (ii) normalizing, (iii) locally confluent, (iv) confluent? Give a brief explanation.

Problem 4 *(Termination, critical pairs)* (10 + 10 = 20 points)

Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{a/0,\ b/0,\ f/2,\ g/1,\ h/2\}$ and let R be the following rewrite system:

$$f(x, f(a, x)) \to h(x, b) \qquad (1)$$
$$f(b, y) \to g(y) \qquad (2)$$
$$h(x, x) \to g(f(a, x)) \qquad (3)$$

Part (a)

Prove the termination of R using a suitable polynomial ordering $\succ$ with the carrier set $\{\, n \in \mathbb{N} \mid n \geq 2 \,\}$ and polynomial coefficients in $\mathbb{N}$.

Part (b)

Compute all critical pairs between rules in R and check whether they are joinable in R.

Problem 5 *(Reduction orderings)* (10 points)

Let $\succ$ be a reduction ordering over $\mathrm{T}_\Sigma(X)$ and let R be a finite set of rewrite rules such that $l \succ r$ holds for each $l \to r \in R$. Prove: For every term $s \in \mathrm{T}_\Sigma(X)$ the set $\{\, t \in \mathrm{T}_\Sigma(X) \mid s \to_R^* t \,\}$ is finite.

Problem 6 *(LPO)* (10 points)

Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{a/0,\ b/0,\ c/0, f/1\}$, let X be a countably infinite set of variables, and let $\succ$ be the LPO with precedence $c > f > b > a$. What can be said about the cardinalities of the following sets of terms?

$$M_1 = \{\, t \in \mathrm{T}_\Sigma(X) \mid t \prec f(f(a)) \,\}$$
$$M_2 = \{\, t \in \mathrm{T}_\Sigma(X) \mid t \prec f(f(c)) \,\}$$
$$M_3 = \{\, t \in \mathrm{T}_\Sigma(X) \mid t \prec f(f(x)) \,\}$$

Give a brief explanation.