Problem 1 *(OBDDs)* (10 points)

Let $\Pi = \{P, Q\}$ be a set of propositional variables; let $P < Q$ be an ordering over Π. Give two Π-formulas F and G and the reduced OBDDs for F, G, $F \wedge G$ and $F \vee G$, such that the reduced OBDDs for F and G have exactly two interior nodes, and the reduced OBDDs for $F \wedge G$ and $F \vee G$ have less than two interior nodes. (interior node = non-leaf node)

Problem 2 *(Algebras and semantics)* (14 points)

Let $\Sigma = (\Omega, \Pi)$ be a signature. For every Σ-formula F without equality let $\mathrm{aneg}(F)$ be the formula that one obtains from F by replacing every atom $p(t_1, \ldots, t_n)$ in F ($p/n \in \Pi$) by its negation $\neg p(t_1, \ldots, t_n)$. Prove: If F is satisfiable, then $\mathrm{aneg}(F)$ is satisfiable.

Problem 3 *(Resolution)* (10 points)

Use the resolution calculus to prove the validity of the following formula:

$$\forall x \exists y \left(p(f(f(x)), y) \wedge \forall z \left(p(f(x), z) \rightarrow p(x, g(x, z)) \right) \right) \rightarrow \forall x \exists y \; p(x, y)$$

Problem 4 *(Tableaux)* (7 + 7 = 14 points)

Check the satisfiability or unsatisfiability of the following formulas by using semantic tableaux. (Use exactly the expansion rules given in the lecture; do not use shortcuts.)

Part (a)

$$\neg\Big((Q \vee \neg P) \rightarrow ((Q \vee P) \rightarrow Q)\Big)$$

Part (b)

$$\Big(\neg Q \rightarrow (P \wedge R)\Big) \wedge \neg\Big((P \vee R) \rightarrow Q\Big)$$

Problem 5 *(Orderings, redundancy)* (4 + 3 + 7 = 14 points)

Let N be the following set of ground clauses:

$$\neg P_3 \lor P_1 \lor P_1 \qquad (1)$$
$$\neg P_2 \lor P_1 \qquad (2)$$
$$P_4 \lor P_4 \qquad (3)$$
$$P_3 \lor \neg P_2 \qquad (4)$$
$$P_4 \lor P_3 \qquad (5)$$

Part (a)

Let the ordering on atoms be defined by $P_4 \succ P_3 \succ P_2 \succ P_1$. Sort the clauses in N with respect to $\succ_{\mathrm{C}}$.

Part (b)

Compute the candidate model $I_N^{\succ}$ for N as described in Section 2.10 of the lecture.

Part (c)

Find another total atom ordering $\succ'$ such that both clause (2) and clause (5) are redundant in N with respect to $\succ'_{\mathrm{C}}$.

Problem 6 *(Herbrand interpretations)* (6 + 6 + 6 = 18 points)

Let $\Sigma = (\Omega, \Pi)$ with $\Omega = \{b/0,\ f/1\}$ and $\Pi = \{p/1\}$.

Part (a)

How many different Herbrand interpretations over Σ do exist? Explain briefly.

Part (b)

How many different Herbrand models over Σ does the formula

$$p(f(f(b))) \ \land\ \forall x \ \big(p(x) \rightarrow p(f(x))\big) \qquad (1)$$

have? Explain briefly.

Part (c)

Every Herbrand model over Σ of (1) is also a model of

$$\forall x\ p(f(f(x))) \qquad (2)$$

Give an example of an algebra that is a model of (1) but not a model of (2).