Universität
des
Saarlandes
FR Informatik

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Tutorials for "Logic in Computer Science"
Exercise sheet 3

Exercise 3.1:
What is the clausal normal form of

$$\exists x\, \forall y\, (\forall z\, p(y,z) \rightarrow (\forall z\, q(y,z) \wedge \neg\, x \approx y))$$

Exercise 3.2:
Let F and G be first-order formulas over Σ, let x be a variable, and let y be a variable not occurring in F, G, and x. Show that

$$\mathcal{A}(\beta)((\exists x\, F) \wedge G) = \mathcal{A}(\beta)(\exists y\, (F[y/x] \wedge G))$$

holds for all Σ-algebras $\mathcal{A}$ and for all assignments $\beta : X \rightarrow \mathcal{A}$.

Exercise 3.3:
Sketch a procedure that decides the validity of prenex formulas $\forall x_1 \dots \forall x_n\, F$ without existential quantifiers, without function symbols, and without equality.

Exercise 3.4:
Derive $\perp$ from the following four clauses using the resolution calculus *Res*:

$$\begin{aligned} p(a) &\vee p(b) \\ \neg p(a) &\vee p(b) \\ p(a) &\vee \neg p(b) \\ \neg p(a) &\vee \neg p(b) \end{aligned}$$

Exercise 3.5:
Let F be a closed first-order formula with equality over a signature $\Sigma = (\Omega, \Pi)$. Let $\sim \notin \Omega$ be a new binary relation symbol (written as an infix operator). Let the set $Eq(\Sigma)$ contain the formulas

$$\forall x\, (x \sim x)$$
$$\forall x, y\, (x \sim y \rightarrow y \sim x)$$
$$\forall x, y, z\, (x \sim y \wedge y \sim z \rightarrow x \sim z)$$

and for every $f/n \in \Omega$ the formula

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n\, (x_1 \sim y_1 \wedge \ldots \wedge x_n \sim y_n \rightarrow f(x_1, \ldots, x_n) \sim f(y_1, \ldots, y_n))$$

and for every $p/n \in \Pi$ the formula

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n\, (x_1 \sim y_1 \wedge \ldots \wedge x_n \sim y_n \wedge p(x_1, \ldots, x_n) \rightarrow p(y_1, \ldots, y_n)).$$

Let $\tilde{F}$ be the formula that one obtains from F if every occurrence of the equality symbol $\approx$ is replaced by the relation symbol $\sim$.

(a) Let $\mathcal{A}$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Show that the interpretation $\sim_{\mathcal{A}}$ of $\sim$ in $\mathcal{A}$ is a congruence relation. (It is enough if you prove *one* of the properties of congruence relations, say symmetry; the other properties are proved analogously.)

(b) Let $\mathcal{A}$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Use the congruence relation $\sim_{\mathcal{A}}$ to construct a model of F and prove that it is a model.

(c) Prove that a formula F is satisfiable if and only if $Eq(\Sigma) \cup \{\tilde{F}\}$ is satisfiable.

Put your solution into the mail box at the door of room 627 in the MPI building (46.1) before May 2, 11:00. Don't forget to write your name and the name of your tutorial group (B, C, D) on your solution.