Problem 1 *(Algebras and semantics)* (3 + 4 + 4 = 11 points)

Let $\Sigma = (\Omega, \Pi)$ be a signature such that Ω contains at least one constant symbol; let

$$\mathcal{A} = (U,\ (f_{\mathcal{A}} : U^n \to U)_{f/n \in \Omega},\ (p_{\mathcal{A}} \subseteq U^m)_{p/m \in \Pi})$$

be a Σ-algebra. An element $a \in U$ is called *term-generated*, if there exists a ground Σ-term t such that $a = \mathcal{A}(\beta)(t)$ for some assignment β. (Note that, if t is ground, then $\mathcal{A}(\beta)(t) = \mathcal{A}(\beta')(t)$ for all assignments β and β'.) The set of all term-generated elements of U is denoted by $\hat{U}$.

Part (a)

Prove: If $f/n \in \Omega$, then $a_1, \dots, a_n \in \hat{U}$ implies $f_{\mathcal{A}}(a_1, \dots, a_n) \in \hat{U}$.

Part (b)

For $f/n \in \Omega$ and $a_1, \dots, a_n \in \hat{U}$ define $f_{\hat{\mathcal{A}}}(a_1, \dots, a_n) = f_{\mathcal{A}}(a_1, \dots, a_n)$; for $p/m \in \Pi$ define $p_{\hat{\mathcal{A}}} = p_{\mathcal{A}} \cap \hat{U}^m$. By part (a),

$$\hat{\mathcal{A}} = (\hat{U},\ (f_{\hat{\mathcal{A}}} : \hat{U}^n \to \hat{U})_{f/n \in \Omega},\ (p_{\hat{\mathcal{A}}} \subseteq \hat{U}^m)_{p/m \in \Pi})$$

is a Σ-algebra. A trivial proof by induction shows that $\mathcal{A}(\beta)(G) = \hat{\mathcal{A}}(\beta)(G)$ for every quantifier-free Σ-formula G and every assignment $\beta : X \to \hat{U}$. Use this result to prove the following proposition: If F is a closed prenex Σ-formula without existential quantifiers, then $\mathcal{A} \models F$ implies $\hat{\mathcal{A}} \models F$.

Part (c)

The property given in part (b) does *not* hold for formulas *with existential quantifiers*. Give an example of a signature Σ, a Σ-algebra $\mathcal{A}$, and a closed prenex Σ-formula F such that $\mathcal{A} \models F$, but $\hat{\mathcal{A}} \not\models F$.

Problem 2 *(Formula transformations, SML)* (6 points)

A quantifier Qx in a formula $Qx\,G$ is superfluous, if all occurrences of x in G are bound in G, or in other words, if there is no occurrence of x in the formula that is bound by Qx itself. For instance, the two underlined quantifiers in the formula

$$\forall y\, \underline{\exists x}\, (p(y) \lor \underline{\forall z}\, \forall x\, q(y, x))$$

are superfluous. Write an SML-function `dropquant : wff -> wff` that takes a formula F as argument and replaces every subformula $Qx\,G$ in F by G if Qx is a superfluous quantifier. You may use the data types and auxiliary functions from the sample solution to tutorial exercise 1.5 (see the appendix).

Problem 3 *(Resolution)* (6 points)

Use the resolution calculus to prove the validity of the following formula:

$$\Big(\forall x\, \forall y\, \big(p(y, f(f(x))) \rightarrow p(y, x)\big)\Big) \rightarrow \Big(\big(\exists x\, \neg p(x, f(x))\big) \vee \big(\forall x\, p(f(x), x)\big)\Big)$$

Problem 4 *(Herbrand interpretations)* (2 + 3 = 5 points)

Let $\Sigma = (\Omega, \Pi)$ with $\Omega = \{b/0,\ c/0\}$ and $\Pi = \{p/1,\ q/2\}$.

Part (a)

How many different Herbrand interpretations over Σ do exist? Explain briefly.

Part (b)

How many different Herbrand models over Σ does the universally quantified clause $\forall x\, (\neg p(c) \vee q(x, b))$ have? Explain briefly.

Problem 5 *(Redundancy)* (8 points)

Prove: If N is a (possibly infinite) set of ground clauses, and every clause in N is redundant in N, then every clause in N is a tautology.

Problem 6 *(Orderings, redundancy)* (2 + 3 + 3 = 8 points)

Let N be the following set of ground clauses:

$$\neg p_2 \vee p_1 \qquad (1)$$
$$\neg p_3 \vee \neg p_1 \qquad (2)$$
$$p_4 \vee p_4 \vee p_1 \qquad (3)$$
$$p_2 \vee \neg p_1 \vee p_1 \qquad (4)$$
$$\neg p_4 \vee \neg p_3 \vee \neg p_2 \qquad (5)$$

Part (a)

Let the ordering on atoms be defined by $p_4 \succ p_3 \succ p_2 \succ p_1$. Sort the clauses in N with respect to $\succ_{\mathrm{C}}$.

Part (b)

Let $\succ$ be defined as in Part (a). Which clauses in N are redundant in N and which are not redundant with respect to $\succ$?

Part (c)

Find another total atom ordering $\succ'$ such that clause (2) is maximal and clause (3) is minimal in N with respect to $\succ'_{\mathrm{C}}$.