Automated Reasoning, 2023/2024 Re-Exam, Sample Solution

Assignment 1

The statement holds.

Proof: Suppose that N contains clauses $C_i \vee D_i$ ($1 \leq i \leq n$) such that $\{\, C_i \mid 1 \leq i \leq n \,\} \models \bot$. Let $\mathcal{A}$ be an arbitrary model of N. We have to show that $\mathcal{A} \models \bigvee_{1 \leq i \leq n} D_i$. Assume otherwise. Then $\mathcal{A}(\bigvee_{1 \leq i \leq n} D_i) = 0$ and therefore $\mathcal{A}(D_i) = 0$ for every $i \in \{1, \ldots, n\}$. On the other hand, $\mathcal{A}$ is a model of every clause in N, and therefore $\mathcal{A}(C_i \vee D_i) = 1$ for every $i \in \{1, \ldots, n\}$. Consequently, $\mathcal{A}(C_i) = 1$ for every $i \in \{1, \ldots, n\}$. This is impossible, however, since $\{\, C_i \mid 1 \leq i \leq n \,\}$ is unsatisfiable.

Grading scheme: Between -3 and -6 points for unclear or wrong quantification over valuations.

Assignment 2

(1) **true:** There are exactly four Herbrand interpretations over Σ, namely $\emptyset$, $\{P(b)\}$, $\{P(c)\}$, and $\{P(b), P(c)\}$, and three of them (the first, the second, and the fourth) are models of $P(b) \vee \neg P(c)$.
(2) **false:** The universe of every Herbrand model is the set of ground terms. Since Σ contains a unary function symbol, there are infinitely many ground terms.
(3) **true:** Take $\mathcal{A}$ with $U_{\mathcal{A}} = \{1, 2, 3, 4, 5\}$, $c_{\mathcal{A}} = 1$, $f_{\mathcal{A}} : n \mapsto 2$, and $P_{\mathcal{A}} = \{2\}$.
(4) **true:** Both formulas are satisfiable, therefore the are equisatisfiable.
(5) **true:** Take an Herbrand interpretation in which all atoms are true; then every clause that has at least one positive literal is true in that interpretation.
(6) **false:** Take $N = \{\bot\}$.
(7) **true:** By Lemma 3.8, every model of $\forall x\, P(f(x))$ is also a model of $\forall y\, P(f(f(y)))$ and thus a model of $\forall y\, P(c) \vee P(f(f(y)))$.

Grading scheme: 4th, 5th, 6th, 7th correct answer: 3 points each.

Assignment 3

Property (1) could be a consequence of Case (2a) or Case (2c) of the LPO definition. Case (2c) can be ruled out, though, since x cannot be larger than $g(y)$. So Case (2a) of the LPO definition must hold, and therefore $f(y) \succ_{\mathrm{lpo}} h(g(y), y)$, which implies $f \succ h$ and $f \succ g$.

Property (2) could be a consequence of Case (2a) or Case (2b) of the LPO definition. Case (2a) can be ruled out, since neither x nor c can be larger than $g(h(x, b))$. So Case (2b) of the LPO definition must hold, and therefore $h \succ g$ and $c \succ b$.

From property (3), we conclude that $g \succ c$ or $b \succ c$. We can rule out $b \succ c$, since we already know $c \succ b$. Therefore $g \succ c$.

Combining these results, we obtain $f \succ h \succ g \succ c \succ b$.

Assignment 4

Part (a) $G_\Sigma(N) = \{P(b)\} \cup \{\neg P(f(f(c))\} \cup \{\, P(f^n(b)) \vee P(f^{n+1}(b)) \mid n \geq 0 \,\} \cup \{\, P(f^n(c)) \vee P(f^{n+1}(c)) \mid n \geq 0 \,\}$.

The clause ordering $\succ_{\mathrm{C}}$ orders $G_\Sigma(N)$ in the following way

$$\begin{array}{rcl}
 & \vdots & \\
\succ_{\mathrm{C}} & P(f^4(b)) \vee \underline{P(f^5(b))} & \\
\succ_{\mathrm{C}} & P(f^4(c)) \vee \underline{P(f^5(c))} & * \\
\succ_{\mathrm{C}} & P(f^3(b)) \vee \underline{P(f^4(b))} & * \\
\succ_{\mathrm{C}} & P(f^3(c)) \vee \underline{P(f^4(c))} & \\
\succ_{\mathrm{C}} & P(f^2(b)) \vee \underline{P(f^3(b))} & \\
\succ_{\mathrm{C}} & P(f^2(c)) \vee \underline{P(f^3(c))} & * \\
\succ_{\mathrm{C}} & P(f(b)) \vee \underline{P(f^2(b))} & * \\
\succ_{\mathrm{C}} & \underline{\neg P(f^2(c)} & \\
\succ_{\mathrm{C}} & P(f(c)) \vee \underline{P(f^2(c))} & \\
\succ_{\mathrm{C}} & P(b) \vee \underline{P(f(b))} & \\
\succ_{\mathrm{C}} & P(c) \vee \underline{P(f(c))} & * \\
\succ_{\mathrm{C}} & \underline{P(b)} & *
\end{array}$$

where the maximal literal in each ground instance is underlined.

Grading scheme: 2 points for determining the ground instances; 5 points for ordering them correctly.

Part (b) The clauses marked with $*$ above are productive and produce their underlined atoms. The candidate interpretation $I^{\succ}_{G_\Sigma(N)}$ is $\{\, P(f^n(b)) \mid n \text{ even} \,\} \cup \{\, P(f^n(c)) \mid n \text{ odd} \,\}$.

Assignment 5

Part (a) By assumption, R is contained in the simplification ordering $\succ$. Futhermore, $f(x)$ is a proper subterm of $f(f(x))$, and since every simplification ordering has the subterm property, we obtain $f(f(x)) \succ f(x)$. Consequently, $R \cup \{f(f(x)) \to f(x)\}$ is also contained in $\succ$ and is therefore terminating.

Since $R \cup \{f(f(x)) \to f(x)\}$ is terminating, it is confluent if and only if it is locally confluent. To prove local confluence, it suffices to show that all critical pairs of $R \cup \{f(f(x)) \to f(x)\}$ are joinable: For all critical pairs between two rules in R, this follows from the fact that R is confluent. The critical pair between $f(f(x)) \to f(x)$ and a renamed copy of it is $f(f(x)) \leftarrow f(f(f(x))) \to f(f(x))$, so it is trivially joinable. Finally, there are no critical pairs between a rule in R and $f(f(x)) \to f(x)$, since f does not occur in any left-hand side of a rule in R.

Note: One cannot prove termination of the combined TRS in the following way: "R is contained in a simplification ordering $\Rightarrow$ R is terminating $\Rightarrow$ $R \cup \{f(f(x)) \to f(x)\}$ is terminating". The reason is that the last implication is invalid: It is not sufficient to assume that R itself is terminating. For instance consider the rewrite system $R = \{g(x,x) \to g(x, f(x))\}$. It is terminating, but not contained in any simplification ordering, and if we add $f(f(x)) \to f(x)$ to R, the resulting rewrite system is not terminating anymore, since $g(f(c), f(c)) \to g(f(c), f(f(c))) \to g(f(c), f(c)) \to \ldots$.

Grading scheme: 5 points for proving termination, 5 points for proving confluence.

Part (b) The TRS $R = \{f(f(b)) \to b\}$ is confluent and contained in every simplification ordering. The combined system $R \cup \{f(f(x)) \to f(x)\}$ is not confluent, though, since $b \leftarrow f(f(b)) \to f(b)$ and b and $f(b)$ are not joinable.

Assignment 6

We start with the given equations (1)–(2).

$$f(g(f(x))) \approx h(x) \tag{1}$$
$$g(h(x)) \approx x \tag{2}$$
$$f(g(f(x))) \to h(x) \tag{3}$$
$$g(h(x)) \to x \tag{4}$$
$$h(g(f(x))) \approx f(g(h(x))) \tag{5}$$
$$h(g(f(x))) \approx f(x) \tag{6}$$
$$h(g(f(x))) \to f(x) \tag{7}$$
$$f(g(f(x))) \approx h(g(h(x))) \tag{8}$$
$$h(x) \approx h(g(h(x))) \tag{9}$$
$$h(x) \approx h(x) \tag{10}$$
$$g(f(x)) \approx g(f(x)) \tag{11}$$

By applying "Orient" twice, we replace (1)–(2) by the corresponding rewrite rules (3)–(4).

Using the critical pair between rule (3) and a renamed copy of itself, the "Deduce" rule adds equation (5). The "Simplify-Eq" rule uses rewrite rule (2) to replace equation (5) by equation (6). The "Orient" rule replaces equation (6) by rule (7).

Using the critical pair between rules (7) and (3), the "Deduce" rule adds equation (8). The "Simplify-Eq" rule uses rewrite rule (3) to replace equation (8) by equation (9). The "Simplify-Eq" rule uses rewrite rule (4) to replace equation (9) by equation (10). Equation (10) is trivial, so it can be eliminated using "Delete".

Using the critical pair between rules (4) and (7), the "Deduce" rule adds equation (11). Equation (11) is again trivial, so it can be eliminated using "Delete".

Since all critical pairs between persisting rules have been computed and all equations have been eliminated, we can stop now; the final rewrite system is $\{(3), (4), (7)\}$.