Automated Reasoning, 2023/2024
Endterm Exam, Sample Solution

Assignment 1

We have to show that N has a model whenever $N \setminus N_0 \setminus N_1$ has a model, and vice versa.

Since $N \setminus N_0 \setminus N_1$ is a subset of N, every model of N is obviously a model of $N \setminus N_0 \setminus N_1$.

For the reverse direction assume that the Σ-algebra $\mathcal{A}$ is a model of $N \setminus N_0 \setminus N_1$. We define a Σ-algebra $\mathcal{B}$ that has the same universe as $\mathcal{A}$ and that agrees with $\mathcal{A}$ for all function and predicate symbols except for $P/1$.

If $Q_{\mathcal{A}} = 1$, we define $P_{\mathcal{B}} = \emptyset$. Since the predicate symbol P does not occur in $N \setminus N_0 \setminus N_1$, $\mathcal{B}$ agrees with $\mathcal{A}$ for all the symbols that occur in these clauses, therefore $\mathcal{B} \models N \setminus N_0 \setminus N_1$. Since all clauses in N_0 contain at least one negated literal $\neg P(t)$ and since $P_{\mathcal{B}}$ is false for every argument, $\mathcal{B} \models N_0$. Finally, all clauses in $N_1 \setminus N_0$ contain the positive literal Q, and since $Q_{\mathcal{B}} = Q_{\mathcal{A}} = 1$, we get $\mathcal{B} \models N_1 \setminus N_0$. Since $N = (N \setminus N_0 \setminus N_1) \cup N_0 \cup (N_1 \setminus N_0)$, we conclude that $\mathcal{B} \models N$.

Otherwise $Q_{\mathcal{A}} = 0$, then we define $P_{\mathcal{B}} = U_{\mathcal{B}}$. Again, for all the symbols that occur in clauses in $N \setminus N_0 \setminus N_1$, $\mathcal{B}$ agrees with $\mathcal{A}$, therefore $\mathcal{B} \models N \setminus N_0 \setminus N_1$. Since all clauses in N_1 contain at least one positive literal $P(t)$ and since $P_{\mathcal{B}}$ is true for every argument, $\mathcal{B} \models N_1$. Finally, all clauses in $N_0 \setminus N_1$ contain the negated literal $\neg Q$, and since $Q_{\mathcal{B}} = Q_{\mathcal{A}} = 0$, we get $\mathcal{B} \models N_1 \setminus N_0$. Since $N = (N \setminus N_0 \setminus N_1) \cup N_1 \cup (N_0 \setminus N_1)$, we conclude again that $\mathcal{B} \models N$.

Grading scheme: 10 points for the "if" part, 2 points for the "only if" part.

Assignment 2

In the example formula, the quantifier $\exists z$ cannot be pushed inside, since the variable z occurs in both parts of the conjunction. The variable x occurs in only one part of the conjunction, but the application of the first miniscoping rule is blocked by the quantifiers $\exists y$ and $\exists z$. Changing the order of several existential quantifiers in front of a subformula, however, yields an equivalent formula. Therefore, the obvious solution is to add a transformation rule that swaps two existential quantifiers in a row, say,

$$H[\exists x\, \exists y\, F]_p \;\Rightarrow_{\mathrm{MS}}\; H[\exists y\, \exists x\, F]_p$$

After applying this rule twice, the quantifier $\exists x$ appears directly before the conjunction, so that now the first miniscoping rule can be applied.

This transformation rule has the drawback, however, that the relation $\Rightarrow_{\mathrm{MS}}$ is no longer terminating. A better approach is to combine the swapping rule and the original miniscoping rule into a single rule, say

$$\begin{aligned} &H[\exists x\, \exists y_1 \ldots \exists y_n\, (F \land G)]_p \\ &\qquad \Rightarrow_{\mathrm{MS}}\; H[\exists y_1 \ldots \exists y_n\, ((\exists x\, F) \land G)]_p \end{aligned}$$

Assignment 3

Part (a) In (1), $P(c,x)$ and $R(g(x),x)$ are not maximal since $P(f(x),x) \succ P(c,x)$ and $P(f(x),x) \succ R(g(x),x)$. In (3), $Q(z)$ is not maximal since $\neg P(z,h(y)) \succ Q(z)$. In (4), $\neg R(g(x),x)$ is not maximal since $Q(x) \succ \neg R(g(x),x)$. The remaining literals are maximal in their clauses: (1)1, (2)1, (3)1, (3)2, (4)1, (4)2, (5)1. This yields the following three inferences:

Res. (1)1, (3)1: mgu: $\{x \mapsto c,\ y \mapsto f(c)\}$
$P(c,c) \lor R(g(c),c) \lor$
$\neg P(z,h(f(c))) \lor Q(z)$

Res. (1)1, (3)2: mgu: $\{x \mapsto h(y),\ z \mapsto f(h(y))\}$
$P(c,h(y)) \lor R(g(h(y)),h(y)) \lor$
$\neg P(y,c) \lor Q(f(h(y)))$

Fact. (4)1, (4)2: mgu: $\{x \mapsto b\}$
$Q(b) \lor \neg R(g(b),b)$

Grading scheme: 2 points for every required inference, 2 points for computing its conclusion correctly; -2 for every unnecessary inference.

Part (b) The conclusion of the first inference above contains the subclause $R(g(c),c)$, which is an instance of clause (5). Therefore, every ground instance of the conclusion follows from a smaller ground instance of (5). Hence the conclusion is redundant.

Assignment 4

Part (a) $f(d) \leftarrow_E f(f(c)) \rightarrow_E f(c) \rightarrow_E d$.

Part (b) The universe of $\mathrm{T}_\Sigma(\emptyset)/E$ consists of the congruence classes of $\mathrm{T}_\Sigma(\emptyset)$ w.r.t. $\leftrightarrow_E^*$. Since every ground term except b and c can be rewritten to d using E, there are three such congruence classes, namely $[b] = \{b\}$, $[c] = \{c\}$, and $[d] = \mathrm{T}_\Sigma(\emptyset) \setminus \{b, c\}$.

Part (c) By Birkhoff's Theorem, an equation $\forall \vec{x}(s \approx t)$ holds in $\mathrm{T}_\Sigma(X)/E$ if and only if $s \leftrightarrow_E^* t$. Therefore, (2) holds in $\mathrm{T}_\Sigma(X)/E$, and (1) and (3) do not hold. (It is not possible to rewrite $f(b)$ to b or $f(x)$ to $f(y)$ using $\leftrightarrow_E$.)

For $\mathcal{T} = \mathrm{T}_\Sigma(\emptyset)/E$, we observe that for every assignment β, $\mathcal{T}(\beta)(f(b)) = [d]$ and $\mathcal{T}(\beta)(b) = [b]$, therefore (1) does not hold in $\mathrm{T}_\Sigma(\emptyset)/E$. On the other hand, for every assignment β, we have $\mathcal{T}(\beta)(f(f(f(y)))) = \mathcal{T}(\beta)(f(f(y))) = [d]$ and $\mathcal{T}(\beta)(f(y)) = \mathcal{T}(\beta)(f(x)) = [d]$, therefore both (2) and (3) hold in $\mathrm{T}_\Sigma(\emptyset)/E$.

Grading scheme: 1 point for each correct answer with a reasonable explanation.

Assignment 5

Part (a) Assume that $s \rightarrow_R t$ using some rewrite rule $l \rightarrow r$ in R. Then $s = s[l\sigma]_p$ and $t = s[r\sigma]_p$. Since $\mathrm{var}(r) \subseteq \mathrm{var}(l)$, we obtain

$$\begin{aligned}\mathrm{var}(t) = \mathrm{var}(s[r\sigma]_p) &\subseteq \mathrm{var}(s) \cup \mathrm{var}(r\sigma) \\ &= \mathrm{var}(s) \cup \textstyle\bigcup_{x \in \mathrm{var}(r)} \mathrm{var}(x\sigma) \\ &\subseteq \mathrm{var}(s) \cup \textstyle\bigcup_{x \in \mathrm{var}(l)} \mathrm{var}(x\sigma) \\ &= \mathrm{var}(s) \cup \mathrm{var}(l\sigma) = \mathrm{var}(s).\end{aligned}$$

Part (b) First note that $s \rightarrow_R^* t$ implies $\mathrm{var}(s) \supseteq \mathrm{var}(t)$; this follows from part (a) by an obvious induction over the length of the rewrite derivation.

Assume that $x \in X$ is a variable, $s \in \mathrm{T}_\Sigma(X)$ is a term such that $x \notin \mathrm{var}(s)$, $R \models x \approx s$, and R is confluent. By Birkhoff's Theorem, $R \models x \approx s$ is equivalent to $x \leftrightarrow_R^* s$. Since confluence is equivalent to the Church-Rosser property, this implies that there exists a term t such that $x \rightarrow_R^* t$ and $s \rightarrow_R^* t$. Now note that the left-hand side of a rewrite rule cannot be a variable; therefore a variable x cannot be rewritten to any other term using $\rightarrow_R$. Consequently, $x = t$. But then $s \rightarrow_R^* x$, which implies that $\mathrm{var}(s) \supseteq \mathrm{var}(x) = \{x\}$, contradicting the assumption that $x \notin \mathrm{var}(s)$.

Assignment 6

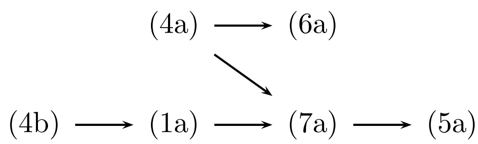
Part (a) The set of defined symbols is $D = \{f, g, h\}$, therefore R has six dependency pairs:

$$f^\sharp(p(x)) \rightarrow h^\sharp(q(x)) \qquad (1a)$$
$$g^\sharp(p(x)) \rightarrow h^\sharp(f(x)) \qquad (4a)$$
$$g^\sharp(p(x)) \rightarrow f^\sharp(x) \qquad (4b)$$
$$g^\sharp(q(g(x))) \rightarrow g^\sharp(b) \qquad (5a)$$
$$h^\sharp(p(x)) \rightarrow g^\sharp(c) \qquad (6a)$$
$$h^\sharp(q(q(x))) \rightarrow g^\sharp(q(x)) \qquad (7a)$$

Note that there is no dependency pair $f^\sharp(f(x)) \rightarrow f^\sharp(x)$ derived from (3), since $f(x)$ is a proper subterm of the left-hand side of (3).

Grading scheme: -1 point for each missing or wrong dependency pair.

Part (b) The approximated dependency graph for R is



As the graph is acyclic, R is terminating.

Grading scheme: 5 points for the dependency graph, -1 point for each missing or incorrect edge, 1 point for showing termination.

Part (c) The exact dependency graph for R contains an edge from a dependency pair $s \rightarrow t$ to a dependency pair $u \rightarrow v$ if $t\sigma \rightarrow_R^* u\tau$ for some instances $t\sigma$ and $u\tau$. For the dependency pairs (4a) and (7a), this condition is not satisfied. Note that rewriting an instance $(h^\sharp(f(x)))\sigma$ using any number of R-steps results either in a term $h^\sharp(f(\ldots))$ or a term $h^\sharp(p(\ldots))$. It is impossible to obtain a term of the form $h^\sharp(q(q(\ldots)))$, that is, an instance of $h^\sharp(q(q(x)))$. Therefore the exact dependency graph has no edge from (4a) to (7a).

Grading scheme: 3 points for determining the correct edge and giving a reasonable explanation.