Automated Reasoning, 2023/2024 Midterm Exam, Sample Solution

Assignment 1

With a reasonable strategy and the given literal selection rule, the CDCL procedure yields

$$\underset{}{\neg P^{\mathrm{d}}}\ \underset{(3)}{\neg Q}\ \neg R^{\mathrm{d}}\ \neg S^{\mathrm{d}}\ \underset{(7)}{T}\ \underset{(8)}{U}\ \underset{(9)}{V} \parallel N$$

At this point, clause (10) is a conflict clause. By resolving (10) with (9), we obtain $\neg T \vee \neg U$ (which is not a backjump clause), by resolving this clause with (8), we obtain $P \vee \neg T$ (11), which is a backjump clause. Using this backjump clause, we remove the last five literals from the trail and add $\neg T^{(11)}$. We continue und obtain

$$\neg P^{\mathrm{d}}\ \underset{(3)}{\neg Q}\ \underset{(11)}{\neg T}\ \underset{(7)}{S}\ \underset{(5)}{\neg R} \parallel N \cup \{(11)\}$$

At this point, clause (6) is a conflict clause. By resolving (6) with (5), we obtain $\neg S \vee T$ (which is not a backjump clause), by resolving this clause with (7), we obtain T (12), which is a backjump clause. Using this backjump clause, we remove all literals from the trail and add $T^{(12)}$. We continue und obtain

$$\underset{(12)}{T}\ \underset{(9)}{V}\ \underset{(10)}{\neg U}\ \underset{(8)}{P}\ \neg Q^{\mathrm{d}}\ \neg R^{\mathrm{d}}\ \neg S^{\mathrm{d}} \parallel N \cup \{(11), (12)\}$$

Since all literals are defined and all clauses in N are true, this is a final state, so by Thm. 2.19, the literals on the trail are a model of N.

Grading scheme: -2 points per error (-1 point, if the decision literal selection strategy was ignored in the last part of the proof).

Assignment 2

Assume that $\mathrm{rep}(F)$ is satisfiable. Then there exists a valuation $\mathcal{A}$ such that $\mathcal{A}(\mathrm{rep}(F)) = 1$. We have to show that there exists a valuation $\mathcal{A}'$ such that $\mathcal{A}'(F) = 1$. Define $\mathcal{A}'$ by $\mathcal{A}'(Q) = \mathcal{A}(R)$ and $\mathcal{A}'(P) = \mathcal{A}(P)$ for every propositional variable $P \in \Pi \setminus \{Q\}$.

We show by induction over the formula structure that $\mathcal{A}'(G) = \mathcal{A}(\mathrm{rep}(G))$ for every Π-formula G.

Case 1: G is a propositional variable. If $G = Q$, then $\mathrm{rep}(Q) = R$. Therefore $\mathcal{A}'(Q) = \mathcal{A}(R) = \mathcal{A}(\mathrm{rep}(Q))$ by definition of $\mathcal{A}'(Q)$. Otherwise $G = P$ for some $P \in \Pi \setminus \{Q\}$, then $\mathrm{rep}(P) = P$. Therefore $\mathcal{A}'(P) = \mathcal{A}(P) = \mathcal{A}(\mathrm{rep}(P))$ by definition of $\mathcal{A}'(P)$.

Case 2: G is a conjunctive formula $G_1 \vee G_2$. We use the induction hypothesis for G_1 and G_2 and obtain $\mathcal{A}'(G) = \mathcal{A}'(G_1 \vee G_2) = \min(\mathcal{A}'(G_1), \mathcal{A}'(G_2)) = \min(\mathcal{A}(\mathrm{rep}(G_1)), \mathcal{A}(\mathrm{rep}(G_2))) = \mathcal{A}(\mathrm{rep}(G_1) \wedge \mathrm{rep}(G_2)) = \mathcal{A}(\mathrm{rep}(G_1 \wedge G_2))$.

Case 3: G is a negation $\neg G_1$. We use the induction hypothesis for G_1 and obtain $\mathcal{A}'(G) = \mathcal{A}'(\neg G_1) = 1 - \mathcal{A}'(G_1) = 1 - \mathcal{A}(\mathrm{rep}(G_1)) = \mathcal{A}(\neg \mathrm{rep}(G_1)) = \mathcal{A}(\mathrm{rep}(\neg G_1))$.

The remaining cases are handled analogously.

Since $\mathcal{A}(\mathrm{rep}(F)) = 1$ by assumption and $\mathcal{A}'(G) = \mathcal{A}(\mathrm{rep}(G))$ for every Π-formula G, we obtain $\mathcal{A}'(F) = 1$, so F is satisfiable.

Assignment 3

Part (a) By assumption, $H[F]_p$ is a valid formula. Therefore it is a satisfiable formula. By Prop. 2.12, it follows that $H[Q]_p \wedge (Q \leftrightarrow F)$ is satisfiable as well.

Part (b) Since Q does not occur in F, it is possible to define a valuation $\mathcal{A}$ such that $\mathcal{A}(Q) \neq \mathcal{A}(F)$. Therefore $\mathcal{A}(Q \leftrightarrow F) = 0$. Since $\mathcal{A}$ is not a model of $H[Q]_p \wedge (Q \leftrightarrow F)$, the formula $H[Q]_p \wedge (Q \leftrightarrow F)$ is not valid.

Grading scheme: 6 points for a correct answer with a correct explanation; typically no points otherwise.

Assignment 4

Let F be the propositional formula

$$((P \leftrightarrow \neg Q) \wedge R) \rightarrow (\neg P \wedge Q).$$

Since there are no trivial subformulas to be eliminated in Step 1 of the algorithm, we start with the introduction of a fresh variable S for $P \leftrightarrow \neg Q$ in Step 2. This subformula occurs in F at a position with negative polarity, therefore we obtain

$$((S \wedge R) \rightarrow (\neg P \wedge Q)) \wedge ((P \leftrightarrow \neg Q) \rightarrow S).$$

The equivalence occurs in the resulting formula at a position with negative polarity, therefore we replace it by a disjunction of conjunctions in Step 3 of the algorithm and obtain

$$\begin{aligned}&((S \wedge R) \rightarrow (\neg P \wedge Q))\\ &\qquad \wedge (((P \wedge \neg Q) \vee (\neg P \wedge \neg\neg Q)) \rightarrow S).\end{aligned}$$

Elimination of implications yields

$$\begin{aligned}&(\neg(S \wedge R) \vee (\neg P \wedge Q))\\ &\qquad \wedge (\neg((P \wedge \neg Q) \vee (\neg P \wedge \neg\neg Q)) \vee S).\end{aligned}$$

After application of De Morgan's law and elimination of multiple negations, we get

$$\begin{aligned}&(\neg S \vee \neg R \vee (\neg P \wedge Q))\\ &\qquad \wedge (((\neg P \vee Q) \wedge (P \vee \neg Q)) \vee S).\end{aligned}$$

Pushing the disjunctions downward, we obtain

$$\begin{aligned}&(\neg S \vee \neg R \vee \neg P)\\ \wedge\ &(\neg S \vee \neg R \vee Q)\\ \wedge\ &(\neg P \vee Q \vee S)\\ \wedge\ &(P \vee \neg Q \vee S),\end{aligned}$$

which is in CNF.

Grading scheme: -2 points per error (-3 points for errors in the polarity-based Tseitin transformation or the polarity-based elimination of equivalences).

Assignment 5

(1) **true:** Since G is unsatisfiable, $\neg G$ is valid. By assumption, there exists some $\mathcal{A}$ such that $\mathcal{A}(F) = 1$; since $\neg G$ is valid, $\mathcal{A}(F \wedge \neg G) = 1$.

(2) **false:** Let $F = \bot$ and $G = \top$. The formula $\top$ is satisfiable and $\bot \models \top$, but $\bot$ is unsatisfiable.

(3) **true:** $\mathcal{A}(F \wedge G) \leq \mathcal{A}(F)$ for every $\mathcal{A}$, therefore $\mathcal{A}(H[F \wedge G]_p) \leq \mathcal{A}(H[F]_p)$ for every $\mathcal{A}$ by Prop. 2.14.

(4) **true:** By definition, we have $\mathcal{A}(G \vee H) = \max(\mathcal{A}(G), \mathcal{A}(H))$, therefore $\mathcal{A}(G \vee H) = 1$ if and only if $\mathcal{A}(G) = 1$ or $\mathcal{A}(H) = 1$.

(5) **false:** Let $F = P \vee Q$, $G = P$, and $H = Q$, then $P \vee Q \models P \vee Q$, but neither $P \vee Q \models P$ nor $P \vee Q \models Q$.

(6) **true:** If $C \in N$ and $\mathcal{A} \models N$, then $\mathcal{A} \models C$ and therefore $\mathcal{A} \models C \vee D$.

(7) **false:** Let $C = \bot$, $D = \bot$, and $N = \{P \vee Q, \neg P \vee \neg Q\}$. Then N is satisfiable, but $N \cup \{\bot\}$ is unsatisfiable.

Grading scheme: 4th, 5th, 6th, 7th correct answer: 3 points each.

Assignment 6

Part (a) The only possible ordering on M is $b \succ a \succ c \succ d$.

Part (b) For rule (4), we need $\{a, a\} \succ_{\mathrm{mul}} \{b, c\}$, therefore $a \succ b$ and $a \succ c$. For rule (5), we need $\{b, b\} \succ_{\mathrm{mul}} \{a, c\}$, therefore $b \succ a$ and $b \succ c$. From $a \succ b$ and $b \succ a$, it follows that $a \succ a$, contradicting irreflexivity.

Part (c) We map every multiset S over M to a pair of two natural numbers, where the first one is $S(a) + S(b)$ (that is, the sum of the numbers of occurrences of a and b in S), and the second one is $S(b)$, and compare these pairs of natural numbers lexicographically. In rule (4), the first component decreases, in rule (5), the first component decreases, in rule (6), the first component remains constant and the second component decreases, therefore the lexicographic combination decreases for all rules (4)–(6).

Alternatively, we can map every multiset S to the natural number $2 \cdot S(a) + 3 \cdot S(b)$. This number also decreases for all rules (4)–(6).