

Path Indexing

Path indexing:

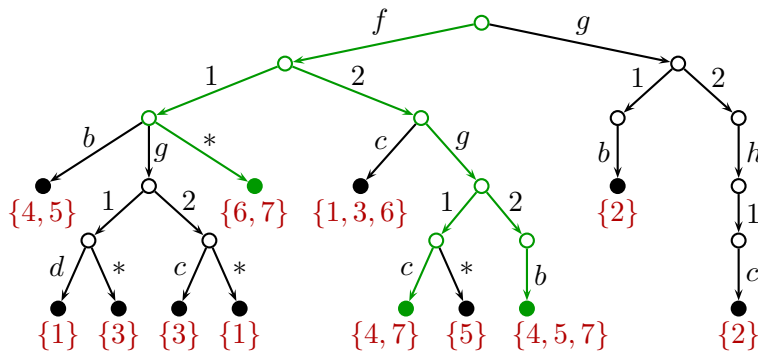
Paths of terms are encoded in a trie (“retrieval tree”).

A star $*$ represents arbitrary variables.

Example: Paths of $f(g(*, b), *)$:
 $f.1.g.1.*$
 $f.1.g.2.b$
 $f.2.*$

Each leaf of the trie contains the set of (pointers to) all terms that contain the respective path.

Example: Path index for $\{f(g(d, *), c), g(b, h(c)), f(g(*, c), c), f(b, g(c, b)), f(b, g(*, b)), f(*, c), f(*, g(c, b))\}$



Advantages:

- Uses little space.
- No backtracking for retrieval.
- Efficient insertion and deletion.
- Good for finding instances.

Disadvantages:

- Retrieval requires combining intermediate results for all paths.

Discrimination Trees

Discrimination trees:

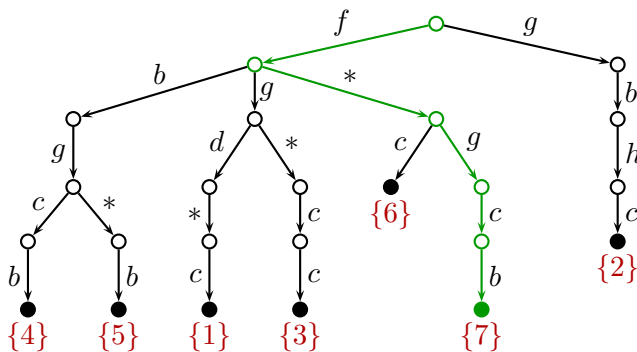
Preorder traversals of terms are encoded in a trie.

A star $*$ represents arbitrary variables.

Example: String of $f(g(*, b), *)$: $f.g.*.b.*$

Each leaf of the trie contains (a pointer to) the term that is represented by the path.

Example: Discrimination tree for $\{f(g(d, *), c), g(b, h(c)), f(g(*, c), c), f(b, g(c, b)), f(b, g(*, b)), f(*, c), f(*, g(c, b))\}$



Advantages:

Each leaf yields one term, hence retrieval does not require intersections of intermediate results for all paths.

Good for finding generalizations.

Disadvantages:

Uses more storage than path indexing (due to less sharing).

Uses still more storage, if jump lists are maintained to speed up the search for instances or unifiable terms.

Feature Vector Indexing

Goal:

C' is subsumed by C if $C' = C\sigma \vee D$.

Find all clauses C' for a given C or vice versa.

If C' is subsumed by C , then

- C' contains at least as many literals as C .
- C' contains at least as many positive literals as C .
- C' contains at least as many negative literals as C .
- C' contains at least as many function symbols as C .
- C' contains at least as many occurrences of f as C .
- C' contains at least as many occurrences of f in negative literals as C .
- the deepest occurrence of f in C' is at least as deep as in C .
- ...

Idea:

Select a list of these “features”.

Compute the “feature vector” (a list of natural numbers) for each clause and store it in a trie.

When searching for a subsuming clause: Traverse the trie, check all clauses for which all features are smaller or equal. (Stop if a subsuming clause is found.)

When searching for subsumed clauses: Traverse the trie, check all clauses for which all features are larger or equal.

Advantages:

Works on the clause level, rather than on the term level.

Specialized for subsumption testing.

Disadvantages:

Needs to be complemented by other index structure for other operations.

Literature

R. Sekar, I. V. Ramakrishnan, and Andrei Voronkov: Term Indexing, Ch. 26 in Robinson and Voronkov (eds.), *Handbook of Automated Reasoning, Vol. II*, Elsevier, 2001.

Stephan Schulz: Simple and Efficient Clause Subsumption with Feature Vector Indexing, in Bonacina and Stickel (eds.), *Automated Reasoning and Mathematics*, LNCS 7788, Springer, 2013.

Christoph Weidenbach: Combining Superposition, Sorts and Splitting, Ch. 27 in Robinson and Voronkov (eds.), *Handbook of Automated Reasoning, Vol. II*, Elsevier, 2001.

7 Outlook

7.1 Satisfiability Modulo Theories (SMT)

CDCL checks satisfiability of propositional formulas.

CDCL can also be used for ground first-order formulas without equality:

Ground first-order atoms are treated like propositional variables.

Truth values of $P(a), Q(a), Q(f(a))$ are independent.

For ground formulas with equality, independence is lost:

If $b \approx c$ is true, then $f(b) \approx f(c)$ must also be true.

Similarly for other theories, e. g. linear arithmetic: $b > 5$ implies $b > 3$.

We can still use CDCL, but we must combine it with a decision procedure for the theory part T :

$M \models_T C$: M and the theory axioms T entail C .

New CDCL rules:

T -Propagate:

$M \parallel N \Rightarrow_{\text{CDCL}(T)} M L \parallel N$

if $M \models_T L$ where L is undefined in M and L or \bar{L} occurs in N .

T -Learn:

$M \parallel N \Rightarrow_{\text{CDCL}(T)} M \parallel N \cup \{C\}$

if $N \models_T C$ and each atom of C occurs in N or M .

T -Backjump:

$M L^d M' \parallel N \cup \{C\} \Rightarrow_{\text{CDCL}(T)} M L' \parallel N \cup \{C\}$

if $M L^d M' \models \neg C$

and there is some “backjump clause” $C' \vee L'$ such that

$N \cup \{C\} \models_T C' \vee L'$ and $M \models \neg C'$,

L' is undefined under M , and

L' or \bar{L}' occurs in N or in $M L^d M'$.

7.2 Sorted Logics

So far, we have considered only unsorted first-order logic.

In practice, one often considers many-sorted logics:

read/2 becomes $read : array \times nat \rightarrow data$.

write/3 becomes $write : array \times nat \times data \rightarrow array$.

Variables: $x : data$

Only one declaration per function/predicate/variable symbol.

All terms, atoms, substitutions must be well-sorted.

Algebras:

Instead of universe $U_{\mathcal{A}}$, one set per sort: $array_{\mathcal{A}}, nat_{\mathcal{A}}$.

Interpretations of function and predicate symbols correspond to their declarations:

$read_{\mathcal{A}} : array_{\mathcal{A}} \times nat_{\mathcal{A}} \rightarrow data_{\mathcal{A}}$

Proof theory, calculi, etc.:

Essentially as in the unsorted case.

More difficult:

Subsorts

Overloading

7.3 Splitting

Tableau-like rule within resolution to eliminate variable-disjoint (positive) disjunctions:

$$\frac{N \cup \{C_1 \vee C_2\}}{N \cup \{C_1\} \quad | \quad N \cup \{C_2\}}$$

if $\text{var}(C_1) \cap \text{var}(C_2) = \emptyset$.

Split clauses are smaller and more likely to be usable for simplification.

Splitting tree is explored using intelligent backtracking.

7.4 Integrating Theories into Resolution

Certain kinds of axioms are

important in practice,

but difficult for theorem provers.

Most important case: equality

but also: orderings, (associativity and) commutativity, ...

Idea: Combine ordered resolution and critical pair computation.

Superposition (ground case):

$$\frac{D' \vee t \approx t' \quad C' \vee s[t] \approx s'}{D' \vee C' \vee s[t'] \approx s'}$$

Superposition (non-ground case):

$$\frac{D' \vee t \approx t' \quad C' \vee s[u] \approx s'}{(D' \vee C' \vee s[t']) \sigma}$$

where $\sigma = \text{mgu}(t, u)$ and u is not a variable.

Advantages:

No variable overlaps (as in KB-completion).

Stronger ordering restrictions:

Only overlaps of (strictly) maximal sides of (strictly) maximal literals are required.

Stronger redundancy criteria.

Similarly for orderings:

Ordered chaining:

$$\frac{D' \vee t' < t \quad C' \vee s < s'}{(D' \vee C' \vee t' < s') \sigma}$$

where σ is a most general unifier of t and s .

Integrating other theories:

Black box:

Use external decision procedure.

Easy, but works only under certain restrictions.

White box:

Integrate using specialized inference rules and theory unification.

Hard work.

Often: integrating more theory axioms is better.

Contents

1	Preliminaries	2
1.1	Mathematical Prerequisites	2
1.2	Abstract Reduction Systems	3
1.3	Orderings	4
1.4	Multisets	8
1.5	Complexity Theory Prerequisites	10
2	Propositional Logic	12
2.1	Syntax	12
2.2	Semantics	15
2.3	Models, Validity, and Satisfiability	16
2.4	Normal Forms	20
2.5	Improving the CNF Transformation	23
2.6	The DPLL Procedure	27
2.7	From DPLL to CDCL	29
2.8	Implementing CDCL	34
2.9	OBDDs	37
2.10	FRAIGs	43
2.11	Other Calculi	43
3	First-Order Logic	44
3.1	Syntax	44
3.2	Semantics	50
3.3	Models, Validity, and Satisfiability	52
3.4	Algorithmic Problems	54
3.5	Normal Forms and Skolemization	55
3.6	Getting Skolem Functions with Small Arity	58
3.7	Herbrand Interpretations	61
3.8	Inference Systems and Proofs	62
3.9	Ground (or propositional) Resolution	64
3.10	Refutational Completeness of Resolution	66
3.11	General Resolution	72
3.12	Theoretical Consequences	80
3.13	Ordered Resolution with Selection	81
3.14	Redundancy	87
3.15	Hyperresolution	91
3.16	Implementing Resolution: The Main Loop	92
3.17	Summary: Resolution Theorem Proving	93
3.18	Semantic Tableaux	93
3.19	Semantic Tableaux for First-Order Logic	100
3.20	Other Deductive Systems	103

4	First-Order Logic with Equality	105
4.1	Handling Equality Naively	105
4.2	Rewrite Systems	106
4.3	Confluence	110
4.4	Critical Pairs	112
4.5	Termination	114
4.6	Knuth-Bendix Completion	122
4.7	Unfailing Completion	127
5	Termination Revisited	130
5.1	Dependency Pairs	130
5.2	Subterm Criterion	132
5.3	Reduction Pairs and Argument Filterings	134
6	Implementing Saturation Procedures	138
6.1	Term Representations	139
6.2	Index Data Structures	139
7	Outlook	145
7.1	Satisfiability Modulo Theories (SMT)	145
7.2	Sorted Logics	146
7.3	Splitting	146
7.4	Integrating Theories into Resolution	147