Confluence and Local Confluence

Theorem 4.11 ("Newman's Lemma") *If a terminating relation $\rightarrow$ is locally confluent, then it is confluent.*

Proof. Let $\rightarrow$ be a terminating and locally confluent relation. Then $\rightarrow^+$ is a well-founded ordering. Define $\phi(a) \;\Leftrightarrow\; \big(\forall b, c : b \leftarrow^* a \rightarrow^* c \Rightarrow b \downarrow c\big)$.

We prove $\phi(a)$ for all $a \in A$ by well-founded induction over $\rightarrow^+$:

Case 1: $b \leftarrow^0 a \rightarrow^* c$: trivial.

Case 2: $b \leftarrow^* a \rightarrow^0 c$: trivial.

Case 3: $b \leftarrow^* b' \leftarrow a \rightarrow c' \rightarrow^* c$: use local confluence, then use the induction hypothesis. □

Rewrite Relations

Corollary 4.12 *If E is convergent (i. e., terminating and confluent), then $s \approx_E t$ if and only if $s \leftrightarrow_E^* t$ if and only if $s{\downarrow_E} = t{\downarrow_E}$.*

Corollary 4.13 *If E is finite and convergent, then $\approx_E$ is decidable.*

Reminder:
If E is terminating, then it is confluent if and only if it is locally confluent.

Problems:

Show local confluence of E.

Show termination of E.

Transform E into an equivalent set of equations that is locally confluent and terminating.

4.4 Critical Pairs

Showing local confluence (Sketch):

Problem: If $t_1 \leftarrow_E t_0 \rightarrow_E t_2$, does there exist a term s such that $t_1 \rightarrow_E^* s \leftarrow_E^* t_2$?

If the two rewrite steps happen in different subtrees (disjoint redexes): yes.

If the two rewrite steps happen below each other (overlap at or below a variable position): yes.

If the left-hand sides of the two rules overlap at a non-variable position: needs further investigation.

Question:
Are there rewrite rules $l_1 \rightarrow r_1$ and $l_2 \rightarrow r_2$ such that some subterm $l_1|_p$ and l_2 have a common instance $(l_1|_p)\sigma_1 = l_2\sigma_2$?

Observation:
If we assume w.l.o.g. that the two rewrite rules do not have common variables, then only a single substitution is necessary: $(l_1|_p)\sigma = l_2\sigma$.

Further observation:
The mgu of $l_1|_p$ and l_2 subsumes all unifiers σ of $l_1|_p$ and l_2.

Let $l_i \rightarrow r_i$ $(i = 1, 2)$ be two rewrite rules in a TRS R whose variables have been renamed such that $\text{var}(l_1) \cap \text{var}(l_2) = \emptyset$. (Remember that $\text{var}(l_i) \supseteq \text{var}(r_i)$.)

Let $p \in \text{pos}(l_1)$ be a position such that $l_1|_p$ is not a variable and σ is an mgu of $l_1|_p$ and l_2.

Then $r_1\sigma \leftarrow l_1\sigma \rightarrow (l_1\sigma)[r_2\sigma]_p$.

$\langle r_1\sigma, (l_1\sigma)[r_2\sigma]_p \rangle$ is called a *critical pair* of R.

The critical pair is *joinable* (or: converges), if $r_1\sigma \downarrow_R (l_1\sigma)[r_2\sigma]_p$.

Theorem 4.14 ("Critical Pair Theorem") *A TRS R is locally confluent if and only if all its critical pairs are joinable.*

Proof. "only if": obvious, since joinability of a critical pair is a special case of local confluence.

"if": Suppose s rewrites to t_1 and t_2 using rewrite rules $l_i \rightarrow r_i \in R$ at positions $p_i \in \text{pos}(s)$, where $i = 1, 2$. Without loss of generality, we can assume that the two rules are variable disjoint, hence $s|_{p_i} = l_i\theta$ and $t_i = s[r_i\theta]_{p_i}$.

We distinguish between two cases: Either p_1 and p_2 are in disjoint subtrees ($p_1 \parallel p_2$), or one is a prefix of the other (w.l.o.g., $p_1 \leq p_2$).

Case 1: $p_1 \parallel p_2$.

Then $s = s[l_1\theta]_{p_1}[l_2\theta]_{p_2}$, and therefore $t_1 = s[r_1\theta]_{p_1}[l_2\theta]_{p_2}$ and $t_2 = s[l_1\theta]_{p_1}[r_2\theta]_{p_2}$.

Let $t_0 = s[r_1\theta]_{p_1}[r_2\theta]_{p_2}$. Then clearly $t_1 \to_R t_0$ using $l_2 \to r_2$ and $t_2 \to_R t_0$ using $l_1 \to r_1$.

Case 2: $p_1 \leq p_2$.

Case 2.1: $p_2 = p_1 q_1 q_2$, where $l_1|_{q_1}$ is some variable x.

In other words, the second rewrite step takes place at or below a variable in the first rule. Suppose that x occurs m times in l_1 and n times in r_1 (where $m \geq 1$ and $n \geq 0$).

Then $t_1 \to_R^* t_0$ by applying $l_2 \to r_2$ at all positions $p_1 q' q_2$, where q' is a position of x in r_1.

Conversely, $t_2 \to_R^* t_0$ by applying $l_2 \to r_2$ at all positions $p_1 q q_2$, where q is a position of x in l_1 different from q_1, and by applying $l_1 \to r_1$ at p_1 with the substitution θ', where $\theta' = \theta[x \mapsto (x\theta)[r_2\theta]_{q_2}]$.

Case 2.2: $p_2 = p_1 p$, where p is a non-variable position of l_1.

Then $s|_{p_2} = l_2\theta$ and $s|_{p_2} = (s|_{p_1})|_p = (l_1\theta)|_p = (l_1|_p)\theta$, so θ is a unifier of l_2 and $l_1|_p$.

Let σ be the mgu of l_2 and $l_1|_p$, then $\theta = \tau \circ \sigma$ and $\langle r_1\sigma, (l_1\sigma)[r_2\sigma]_p \rangle$ is a critical pair.

By assumption, it is joinable, so $r_1\sigma \to_R^* v \leftarrow_R^* (l_1\sigma)[r_2\sigma]_p$.

Consequently, $t_1 = s[r_1\theta]_{p_1} = s[r_1\sigma\tau]_{p_1} \to_R^* s[v\tau]_{p_1}$ and $t_2 = s[r_2\theta]_{p_2} = s[(l_1\theta)[r_2\theta]_p]_{p_1} = s[(l_1\sigma\tau)[r_2\sigma\tau]_p]_{p_1} = s[((l_1\sigma)[r_2\sigma]_p)\tau]_{p_1} \to_R^* s[v\tau]_{p_1}$.

This completes the proof of the Critical Pair Theorem. □

Note: Critical pairs between a rule and (a renamed variant of) itself must be considered – except if the overlap is at the root (i. e., $p = \varepsilon$).

Corollary 4.15 *A terminating TRS R is confluent if and only if all its critical pairs are joinable.*

Proof. By Newman's Lemma and the Critical Pair Theorem. □

Corollary 4.16 *For a finite terminating TRS, confluence is decidable.*

Proof. For every pair of rules and every non-variable position in the first rule there is at most one critical pair $\langle u_1, u_2 \rangle$.

Reduce every u_i to some normal form u_i'. If $u_1' = u_2'$ for every critical pair, then R is confluent, otherwise there is some non-confluent situation $u_1' \leftarrow_R^* u_1 \leftarrow_R s \to_R u_2 \to_R^* u_2'$. □

4.5 Termination

Termination problems:

Given a finite TRS R and a term t, are all R-reductions starting from t terminating?

Given a finite TRS R, are all R-reductions terminating?

Proposition 4.17 *Both termination problems for TRSs are undecidable in general.*

Proof. Encode Turing machines using rewrite rules and reduce the (uniform) halting problems for TMs to the termination problems for TRSs. □

Consequence:

Decidable criteria for termination are not complete.

Two Different Scenarios

Depending on the application, the TRS whose termination we want to show can be

(i) fixed and known in advance, or

(ii) evolving (e.g., generated by some saturation process).

Methods for case (ii) are also usable for case (i).
Many methods for case (i) are not usable for case (ii).

We will first consider case (ii);
additional techniques for case (i) will be considered later.

Reduction Orderings

Goal:

Given a finite TRS R, show termination of R by looking at finitely many rules $l \to r \in R$, rather than at infinitely many possible replacement steps $s \to_R s'$.

A binary relation $\sqsupset$ over $\mathrm{T}_\Sigma(X)$ is called *compatible with* Σ*-operations*, if $s \sqsupset s'$ implies $f(t_1, \dots, s, \dots, t_n) \sqsupset f(t_1, \dots, s', \dots, t_n)$ for all $f \in \Omega$ and $s, s', t_i \in \mathrm{T}_\Sigma(X)$.

Lemma 4.18 *The relation* $\sqsupset$ *is compatible with* Σ*-operations, if and only if* $s \sqsupset s'$ *implies* $t[s]_p \sqsupset t[s']_p$ *for all* $s, s', t \in \mathrm{T}_\Sigma(X)$ *and* $p \in \mathrm{pos}(t)$.

Note: *compatible with Σ-operations* = *compatible with contexts.*

A binary relation $\sqsupset$ over $\mathrm{T}_\Sigma(X)$ is called *stable under substitutions,* if $s \sqsupset s'$ implies $s\sigma \sqsupset s'\sigma$ for all $s, s' \in \mathrm{T}_\Sigma(X)$ and substitutions σ.

A binary relation $\sqsupset$ is called a *rewrite relation,* if it is compatible with Σ-operations and stable under substitutions.

Example: If R is a TRS, then $\to_R$ is a rewrite relation.

A strict partial ordering over $\mathrm{T}_\Sigma(X)$ that is a rewrite relation is called *rewrite ordering.*

A well-founded rewrite ordering is called *reduction ordering.*

Theorem 4.19 *A TRS R terminates if and only if there exists a reduction ordering $\succ$ such that $l \succ r$ for every rule $l \to r \in R$.*

Proof. "if": $s \to_R s'$ if and only if $s = t[l\sigma]_p$, $s' = t[r\sigma]_p$. If $l \succ r$, then $l\sigma \succ r\sigma$ and therefore $t[l\sigma]_p \succ t[r\sigma]_p$. This implies $\to_R \subseteq \succ$. Since $\succ$ is a well-founded ordering, $\to_R$ is terminating.

"only if": Define $\succ\ =\ \to_R^+$. If $\to_R$ is terminating, then $\succ$ is a reduction ordering. □

The Interpretation Method

Proving termination by interpretation:

Let $\mathcal{A}$ be a Σ-algebra; let $\succ$ be a well-founded strict partial ordering on its universe.

Define the ordering $\succ_\mathcal{A}$ over $\mathrm{T}_\Sigma(X)$ by $s \succ_\mathcal{A} t$ iff $\mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(t)$ for all assignments $\beta : X \to U_\mathcal{A}$.

Is $\succ_\mathcal{A}$ a reduction ordering?

Lemma 4.20 *$\succ_\mathcal{A}$ is stable under substitutions.*

Proof. Let $s \succ_\mathcal{A} s'$, that is, $\mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(s')$ for all assignments $\beta : X \to U_\mathcal{A}$. Let σ be a substitution. We have to show that $\mathcal{A}(\gamma)(s\sigma) \succ \mathcal{A}(\gamma)(s'\sigma)$ for all assignments $\gamma : X \to U_\mathcal{A}$. Choose $\beta = \gamma \circ \sigma$, then by the substitution lemma, $\mathcal{A}(\gamma)(s\sigma) = \mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(s') = \mathcal{A}(\gamma)(s'\sigma)$. Therefore $s\sigma \succ_\mathcal{A} s'\sigma$. □

A function $f : U_\mathcal{A}^n \to U_\mathcal{A}$ is called *monotone* (with respect to $\succ$), if $a \succ a'$ implies $f(b_1, \dots, a, \dots, b_n) \succ f(b_1, \dots, a', \dots, b_n)$ for all $a, a', b_i \in U_\mathcal{A}$.

Lemma 4.21 *If the interpretation $f_\mathcal{A}$ of every function symbol f is monotone w. r. t. $\succ$, then $\succ_\mathcal{A}$ is compatible with Σ-operations.*

Proof. Let $s \succ_{\mathcal{A}} s'$, that is, $\mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(s')$ for all $\beta : X \to U_{\mathcal{A}}$. Let $\beta : X \to U_{\mathcal{A}}$ be an arbitrary assignment. Then

$$\begin{aligned}\mathcal{A}(\beta)(f(t_1,\dots,s,\dots,t_n)) &= f_{\mathcal{A}}(\mathcal{A}(\beta)(t_1),\dots,\mathcal{A}(\beta)(s),\dots,\mathcal{A}(\beta)(t_n))\\ &\succ f_{\mathcal{A}}(\mathcal{A}(\beta)(t_1),\dots,\mathcal{A}(\beta)(s'),\dots,\mathcal{A}(\beta)(t_n))\\ &= \mathcal{A}(\beta)(f(t_1,\dots,s',\dots,t_n))\end{aligned}$$

Therefore $f(t_1,\dots,s,\dots,t_n) \succ_{\mathcal{A}} f(t_1,\dots,s',\dots,t_n)$. □

Theorem 4.22 *If the interpretation $f_{\mathcal{A}}$ of every function symbol f is monotone w. r. t. $\succ$, then $\succ_{\mathcal{A}}$ is a reduction ordering.*

Proof. By the previous two lemmas, $\succ_{\mathcal{A}}$ is a rewrite relation. If there were an infinite chain $s_1 \succ_{\mathcal{A}} s_2 \succ_{\mathcal{A}} \dots$, then it would correspond to an infinite chain $\mathcal{A}(\beta)(s_1) \succ \mathcal{A}(\beta)(s_2) \succ \dots$ (with β chosen arbitrarily). Thus $\succ_{\mathcal{A}}$ is well-founded. Irreflexivity and transitivity are proved similarly. □

Polynomial Orderings

Polynomial orderings:

Instance of the interpretation method:

The carrier set $U_{\mathcal{A}}$ is $\mathbb{N}$ or some subset of $\mathbb{N}$.

To every function symbol f/n we associate a polynomial $P_f(X_1,\dots,X_n) \in \mathbb{N}[X_1,\dots,X_n]$ with coefficients in $\mathbb{N}$ and indeterminates $X_1,\dots,X_n$. Then we define $f_{\mathcal{A}}(a_1,\dots,a_n) = P_f(a_1,\dots,a_n)$ for $a_i \in U_{\mathcal{A}}$.

Requirement 1:

If $a_1,\dots,a_n \in U_{\mathcal{A}}$, then $f_{\mathcal{A}}(a_1,\dots,a_n) \in U_{\mathcal{A}}$. (Otherwise, $\mathcal{A}$ would not be a Σ-algebra.)

Requirement 2:

$f_{\mathcal{A}}$ must be monotone (w. r. t. $\succ$).

From now on:

$U_{\mathcal{A}} = \{\, n \in \mathbb{N} \mid n \geq 1 \,\}$.

If arity$(f) = 0$, then P_f is a constant ≥ 1.

If arity$(f) = n \geq 1$, then P_f is a polynomial $P(X_1,\dots,X_n)$, such that every X_i occurs in some monomial $m \cdot X_1^{j_1} \cdots X_k^{j_k}$ with exponent at least 1 and non-zero coefficient $m \in \mathbb{N}$.

⇒ Requirements 1 and 2 are satisfied.

The mapping from function symbols to polynomials can be extended to terms: A term t containing the variables $x_1, \dots, x_n$ yields a polynomial P_t with indeterminates $X_1, \dots, X_n$ (where X_i corresponds to $\beta(x_i)$).

Example:

$\Omega = \{b/0,\ f/1,\ g/3\}$
$P_b = 3, \quad P_f(X_1) = X_1^2, \quad P_g(X_1, X_2, X_3) = X_1 + X_2 X_3.$

Let $t = g(f(b), f(x), y)$, then $P_t(X, Y) = 9 + X^2 Y$.

If P, Q are polynomials in $\mathbb{N}[X_1, \dots, X_n]$, we write $P > Q$ if $P(a_1, \dots, a_n) > Q(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in U_{\mathcal{A}}$.

Clearly, $s \succ_{\mathcal{A}} t$ iff $P_s > P_t$ iff $P_s - P_t > 0$.

Question: Can we check $P_s - P_t > 0$ automatically?

Hilbert's 10th Problem:

Given a polynomial $P \in \mathbb{Z}[X_1, \dots, X_n]$ with integer coefficients, is $P = 0$ for some n-tuple of natural numbers?

Theorem 4.23 *Hilbert's 10th Problem is undecidable.*

Proposition 4.24 *Given a polynomial interpretation and two terms s, t, it is undecidable whether $P_s > P_t$.*

Proof. By reduction of Hilbert's 10th Problem. □

One easy case:

If we restrict to linear polynomials, deciding whether $P_s - P_t > 0$ is trivial:

$\sum k_i a_i + k > 0$ for all $a_1, \dots, a_n \geq 1$ if and only if

$k_i \geq 0$ for all $i \in \{1, \dots, n\}$,

and $\sum k_i + k > 0$

Another possible solution:

Test whether $P_s(a_1, \dots, a_n) > P_t(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in \{\, x \in \mathbb{R} \mid x \geq 1 \,\}$.

This is decidable (but hard). Since $U_{\mathcal{A}} \subseteq \{\, x \in \mathbb{R} \mid x \geq 1 \,\}$, it implies $P_s > P_t$.

Alternatively:

Use fast overapproximations.

Simplification Orderings

The *proper subterm ordering* $\rhd$ is defined by $s \rhd t$ if and only if $s|_p = t$ for some position $p \neq \varepsilon$ of s.

A rewrite ordering $\succ$ over $\mathrm{T}_\Sigma(X)$ is called *simplification ordering,* if it has the *subterm property:* $s \rhd t$ implies $s \succ t$ for all $s, t \in \mathrm{T}_\Sigma(X)$.

Example:

Let R_{emb} be the rewrite system $R_{\text{emb}} = \{\, f(x_1, \ldots, x_n) \to x_i \mid f/n \in \Omega,\ 1 \leq i \leq n \,\}$.

Define $\rhd_{\text{emb}} = \to^+_{R_{\text{emb}}}$ and $\trianglerighteq_{\text{emb}} = \to^*_{R_{\text{emb}}}$ *("homeomorphic embedding relation").*

$\rhd_{\text{emb}}$ is a simplification ordering.

Lemma 4.25 *If $\succ$ is a simplification ordering, then $s \rhd_{\text{emb}} t$ implies $s \succ t$ and $s \trianglerighteq_{\text{emb}} t$ implies $s \succeq t$.*

Proof. Since $\succ$ is transitive and $\succeq$ is transitive and reflexive, it suffices to show that $s \to_{R_{\text{emb}}} t$ implies $s \succ t$. By definition, $s \to_{R_{\text{emb}}} t$ if and only if $s = s[l\sigma]$ and $t = s[r\sigma]$ for some rule $l \to r \in R_{\text{emb}}$. Obviously, $l \rhd r$ for all rules in R_{emb}, hence $l \succ r$. Since $\succ$ is a rewrite relation, $s = s[l\sigma] \succ s[r\sigma] = t$. □

Goal:

Show that every simplification ordering is well-founded (and therefore a reduction ordering).

Note: This works only for *finite* signatures!

To fix this for infinite signatures, the definition of simplification orderings and the definition of embedding have to be modified.

Theorem 4.26 ("Kruskal's Theorem") *Let Σ be a finite signature, let X be a finite set of variables. Then for every infinite sequence $t_1, t_2, t_3, \ldots$ there are indices $j > i$ such that $t_j \trianglerighteq_{\text{emb}} t_i$. ($\trianglerighteq_{\text{emb}}$ is called* a well-partial-ordering (wpo).*)*

Proof. See Baader and Nipkow, page 113–115. □

Theorem 4.27 (Dershowitz) *If Σ is a finite signature, then every simplification ordering $\succ$ on $\mathrm{T}_\Sigma(X)$ is well-founded (and therefore a reduction ordering).*

Proof. Suppose that $t_1 \succ t_2 \succ t_3 \succ \dots$ is an infinite descending chain.

First assume that there is an $x \in \mathrm{var}(t_{i+1}) \setminus \mathrm{var}(t_i)$. Let $\sigma = \{x \mapsto t_i\}$, then $t_{i+1}\sigma \trianglerighteq x\sigma = t_i$ and therefore $t_i = t_i\sigma \succ t_{i+1}\sigma \succeq t_i$, contradicting reflexivity.

Consequently, $\mathrm{var}(t_i) \supseteq \mathrm{var}(t_{i+1})$ and $t_i \in \mathrm{T}_\Sigma(V)$ for all i, where V is the finite set $\mathrm{var}(t_1)$. By Kruskal's Theorem, there are $i < j$ with $t_i \trianglelefteq_{\mathrm{emb}} t_j$. Hence $t_i \preceq t_j$, contradicting $t_i \succ t_j$. □

There are reduction orderings that are not simplification orderings and terminating TRSs that are not contained in any simplification ordering.

Example:

Let $R = \{f(f(x)) \to f(g(f(x)))\}$.

R terminates and $\to_R^+$ is therefore a reduction ordering.

Assume that $\to_R$ were contained in a simplification ordering $\succ$. Then $f(f(x)) \to_R f(g(f(x)))$ implies $f(f(x)) \succ f(g(f(x)))$, and $f(g(f(x))) \trianglerighteq_{\mathrm{emb}} f(f(x))$ implies $f(g(f(x))) \succeq f(f(x))$, hence $f(f(x)) \succ f(f(x))$.

Path Orderings

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω.

The *lexicographic path ordering* $\succ_{\mathrm{lpo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{\mathrm{lpo}} t$ iff

(1) $t \in \mathrm{var}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and

- (a) $s_i \succeq_{\mathrm{lpo}} t$ for some i, or
- (b) $f \succ g$ and $s \succ_{\mathrm{lpo}} t_j$ for all j, or
- (c) $f = g$, $s \succ_{\mathrm{lpo}} t_j$ for all j, and $(s_1, \dots, s_m)\ (\succ_{\mathrm{lpo}})_{\mathrm{lex}}\ (t_1, \dots, t_n)$.

Lemma 4.28 $s \succ_{\mathrm{lpo}} t$ *implies* $\mathrm{var}(s) \supseteq \mathrm{var}(t)$.

Proof. By induction on $|s| + |t|$ and case analysis. □

Theorem 4.29 *$\succ_{\text{lpo}}$ is a simplification ordering on $\mathrm{T}_\Sigma(X)$.*

Proof. Show transitivity, subterm property, stability under substitutions, compatibility with Σ-operations, and irreflexivity, usually by induction on the sum of the term sizes and case analysis. Details: Baader and Nipkow, page 119/120. □

Theorem 4.30 *If the precedence $\succ$ is total, then the lexicographic path ordering $\succ_{\text{lpo}}$ is total on ground terms, i. e., for all $s, t \in \mathrm{T}_\Sigma(\emptyset)$: $s \succ_{\text{lpo}} t \vee t \succ_{\text{lpo}} s \vee s = t$.*

Proof. By induction on $|s| + |t|$ and case analysis. □

Recapitulation:

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω. The *lexicographic path ordering* $\succ_{\text{lpo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{\text{lpo}} t$ iff

(1) $t \in \mathrm{var}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

  (a) $s_i \succeq_{\text{lpo}} t$ for some i, or

  (b) $f \succ g$ and $s \succ_{\text{lpo}} t_j$ for all j, or

  (c) $f = g$, $s \succ_{\text{lpo}} t_j$ for all j, and $(s_1, \ldots, s_m)\ (\succ_{\text{lpo}})_{\text{lex}}\ (t_1, \ldots, t_n)$.

There are several possibilities to compare subterms in (2)(c):

- compare list of subterms lexicographically left-to-right (*"lexicographic path ordering (lpo)"*, Kamin and Lévy)
- compare list of subterms lexicographically right-to-left (or according to some permutation π)
- compare multiset of subterms using the multiset extension (*"multiset path ordering (mpo)"*, Dershowitz)
- to each function symbol $f/n \in \Omega$ with $n \geq 1$ associate a status $\in \{mul\} \cup \{\, lex_\pi \mid \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \,\}$ and compare according to that status (*"recursive path ordering (rpo) with status"*)

The Knuth-Bendix Ordering

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω, let $w : \Omega \cup X \to \mathbb{R}_0^+$ be a *weight function*, such that the following admissibility conditions are satisfied:

$w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in X$; $w(c) \geq w_0$ for all constants $c \in \Omega$.

If $w(f) = 0$ for some $f/1 \in \Omega$, then $f \succ g$ for all $g/n \in \Omega$ with $f \neq g$.

The weight function w can be extended to terms recursively:

$$w(f(t_1, \dots, t_n)) = w(f) + \sum_{1 \leq i \leq n} w(t_i)$$

or alternatively

$$w(t) = \sum_{x \in \mathrm{var}(t)} w(x) \cdot \#(x, t) + \sum_{f \in \Omega} w(f) \cdot \#(f, t).$$

where $\#(a, t)$ is the number of occurrences of a in t.

The *Knuth-Bendix ordering* $\succ_{\mathrm{kbo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ and w is defined by: $s \succ_{\mathrm{kbo}} t$ iff

(1) $\#(x, s) \geq \#(x, t)$ for all variables x and $w(s) > w(t)$, or

(2) $\#(x, s) \geq \#(x, t)$ for all variables x, $w(s) = w(t)$, and

  (a) $t = x$, $s = f^n(x)$ for some $n \geq 1$, or

  (b) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and $f \succ g$, or

  (c) $s = f(s_1, \dots, s_m)$, $t = f(t_1, \dots, t_m)$, and $(s_1, \dots, s_m)\ (\succ_{\mathrm{kbo}})_{\mathrm{lex}}\ (t_1, \dots, t_m)$.

Theorem 4.31 *The Knuth-Bendix ordering induced by $\succ$ and w is a simplification ordering on $\mathrm{T}_\Sigma(X)$.*

Proof. Baader and Nipkow, pages 125–129. □

Remark

If $\Pi \neq \emptyset$, then all the term orderings described in this section can also be used to compare non-equational atoms by treating predicate symbols like function symbols.

4.6 Knuth-Bendix Completion

Completion:

Goal: Given a set E of equations, transform E into an equivalent convergent set R of rewrite rules.
(If R is finite: decision procedure for E.)

Knuth-Bendix Completion: Idea

How to ensure termination?

Fix a reduction ordering $\succ$ and construct R in such a way that $\rightarrow_R \subseteq \succ$ (i. e., $l \succ r$ for every $l \rightarrow r \in R$).

How to ensure confluence?

Check that all critical pairs are joinable.

Note: Every critical pair $\langle s, t \rangle$ can be *made* joinable by adding $s \rightarrow t$ or $t \rightarrow s$ to R.

(Actually, we first add $s \approx t$ to E and later try to turn it into a rule that is contained in $\succ$; this gives us some additional degree of freedom.)

Knuth-Bendix Completion: Inference Rules

The completion procedure is presented as a set of inference rules working on a set of equations E and a set of rules R: $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$

At the beginning, $E = E_0$ is the input set and $R = R_0$ is empty. At the end, E should be empty; then R is the result.

For each step $E, R \vdash E', R'$, the equational theories of $E \cup R$ and $E' \cup R'$ agree: $\approx_{E \cup R} = \approx_{E' \cup R'}$.

Notations:

The formula $s \mathrel{\dot{\approx}} t$ denotes either $s \approx t$ or $t \approx s$.

$\mathrm{CP}(R)$ denotes the set of all critical pairs between rules in R.

Orient:

$$\frac{E \cup \{s \mathrel{\dot{\approx}} t\}, \quad R}{E, \quad R \cup \{s \to t\}} \qquad \text{if } s \succ t$$

Note: There are equations $s \approx t$ that cannot be oriented, i. e., neither $s \succ t$ nor $t \succ s$.

Trivial equations cannot be oriented – but we don't need them anyway:

Delete:

$$\frac{E \cup \{s \approx s\}, \quad R}{E, \quad R}$$

Critical pairs between rules in R are turned into additional equations:

Deduce:

$$\frac{E, \quad R}{E \cup \{s \approx t\}, \quad R} \qquad \text{if } \langle s, t \rangle \in \mathrm{CP}(R).$$

Note: If $\langle s, t \rangle \in \mathrm{CP}(R)$ then $s \leftarrow_R u \rightarrow_R t$ and hence $R \models s \approx t$.

The following inference rules are not absolutely necessary, but very useful (e. g., to get rid of joinable critical pairs and to deal with equations that cannot be oriented):

Simplify-Eq:

$$\frac{E \cup \{s \mathrel{\dot{\approx}} t\}, \quad R}{E \cup \{u \approx t\}, \quad R} \qquad \text{if } s \rightarrow_R u.$$

Simplification of the right-hand side of a rule is unproblematic:

R-Simplify-Rule:

$$\frac{E, \quad R \cup \{s \to t\}}{E, \quad R \cup \{s \to u\}} \qquad \text{if } t \rightarrow_R u.$$

Simplification of the left-hand side may influence orientability and orientation. Therefore, it yields an *equation*:

L-Simplify-Rule:

$$\frac{E, \quad R \cup \{s \to t\}}{E \cup \{u \approx t\}, \quad R} \qquad \begin{array}{l}\text{if } s \rightarrow_R u \text{ using a rule } l \to r \in R \\ \text{such that } s \sqsupset l \text{ (see below).}\end{array}$$

For technical reasons, the lhs of $s \to t$ may only be simplified using a rule $l \to r$, if $l \to r$ *cannot* be simplified using $s \to t$, that is, if $s \sqsupset l$, where the *encompassment quasi-ordering* $\sqsupseteq$ is defined by

$$s \sqsupseteq l \ \text{ if } \ s|_p = l\sigma \text{ for some } p \text{ and } \sigma$$

and $\sqsupset = \sqsupseteq \setminus \sqsubseteq$ is the strict part of $\sqsupseteq$.

Lemma 4.32 *$\sqsupset$ is a well-founded strict partial ordering.*

Lemma 4.33 *If $E, R \vdash E', R'$, then $\approx_{E \cup R} = \approx_{E' \cup R'}$.*

Lemma 4.34 *If $E, R \vdash E', R'$ and $\to_R \subseteq \succ$, then $\to_{R'} \subseteq \succ$.*

Note: Like in ordered resolution, simplification should be preferred to deduction:

- Simplify/delete whenever possible.
- Otherwise, orient an equation, if possible.
- Last resort: compute critical pairs.

Knuth-Bendix Completion: Correctness Proof

If we run the completion procedure on a set E of equations, different things can happen:

(1) We reach a state where no more inference rules are applicable and E is not empty.
$\Rightarrow$ Failure (try again with another ordering?)

(2) We reach a state where E is empty and all critical pairs between the rules in the current R have been checked.

(3) The procedure runs forever.

In order to treat these cases simultaneously, we need some definitions.

A (finite or infinite sequence) $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$ with $R_0 = \emptyset$ is called a *run* of the completion procedure with input E_0 and $\succ$.

For a run, $E_\infty = \bigcup_{i \geq 0} E_i$ and $R_\infty = \bigcup_{i \geq 0} R_i$.

The sets of *persistent equations or rules* of the run are $E_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} E_j$ and $R_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} R_j$.

Note: If the run is finite and ends with E_n, R_n, then $E_* = E_n$ and $R_* = R_n$.

A run is called *fair*, if $CP(R_*) \subseteq E_\infty$ (i. e., if every critical pair between persisting rules is computed at some step of the derivation).

Goal:

Show: If a run is fair and E_* is empty, then R_* is convergent and equivalent to E_0.

In particular: If a run is fair and E_* is empty, then $\approx_{E_0} = \approx_{E_\infty \cup R_\infty} = \leftrightarrow^*_{E_\infty \cup R_\infty} = \downarrow_{R_*}$.

General assumptions from now on:

$E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$ is a fair run.

R_0 and E_* are empty.

A *proof* of $s \approx t$ in $E_\infty \cup R_\infty$ is a finite sequence $(s_0, \ldots, s_n)$ such that $s = s_0$, $t = s_n$, and for all $i \in \{1, \ldots, n\}$:

(1) $s_{i-1} \leftrightarrow_{E_\infty} s_i$, or

(2) $s_{i-1} \rightarrow_{R_\infty} s_i$, or

(3) $s_{i-1} \leftarrow_{R_\infty} s_i$.

The pairs (s_{i-1}, s_i) are called *proof steps.*

A proof is called a *rewrite proof in* R_*, if there is a $k \in \{0, \ldots, n\}$ such that $s_{i-1} \rightarrow_{R_*} s_i$ for $1 \leq i \leq k$ and $s_{i-1} \leftarrow_{R_*} s_i$ for $k + 1 \leq i \leq n$

Idea (Bachmair, Dershowitz, Hsiang):

Define a well-founded ordering on proofs, such that for every proof that is not a rewrite proof in R_* there is an equivalent smaller proof.

Consequence: For every proof there is an equivalent rewrite proof in R_*.

We associate a *cost* $c(s_{i-1}, s_i)$ with every proof step as follows:

(1) If $s_{i-1} \leftrightarrow_{E_\infty} s_i$, then $c(s_{i-1}, s_i) = (\{s_{i-1}, s_i\}, -, -)$, where the first component is a multiset of terms and $-$ denotes an arbitrary (irrelevant) term.

(2) If $s_{i-1} \rightarrow_{R_\infty} s_i$ using $l \rightarrow r$, then $c(s_{i-1}, s_i) = (\{s_{i-1}\}, l, s_i)$.

(3) If $s_{i-1} \leftarrow_{R_\infty} s_i$ using $l \rightarrow r$, then $c(s_{i-1}, s_i) = (\{s_i\}, l, s_{i-1})$.

Proof steps are compared using the lexicographic combination of the multiset extension of the reduction ordering $\succ$, the encompassment ordering $\sqsupset$, and the reduction ordering $\succ$.

The cost $c(P)$ of a proof P is the multiset of the costs of its proof steps.

The *proof ordering* $\succ_C$ compares the costs of proofs using the multiset extension of the proof step ordering.

Lemma 4.35 *$\succ_C$ is a well-founded ordering.*

Lemma 4.36 *Let P be a proof in $E_\infty \cup R_\infty$. If P is not a rewrite proof in R_*, then there exists an equivalent proof P' in $E_\infty \cup R_\infty$ such that $P \succ_C P'$.*

Proof. If P is not a rewrite proof in R_*, then it contains

(a) a proof step that is in E_∞, or
(b) a proof step that is in $R_\infty \setminus R_*$, or
(c) a subproof $s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1}$ (peak).

We show that in all three cases the proof step or subproof can be replaced by a smaller subproof:

Case (a): A proof step using an equation $s \dot{\approx} t$ is in E_∞. This equation must be deleted during the run.

If $s \dot{\approx} t$ is deleted using *Orient*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_i \dots \quad \Longrightarrow \quad \dots s_{i-1} \rightarrow_{R_\infty} s_i \dots$$

If $s \dot{\approx} t$ is deleted using *Delete*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_{i-1} \dots \quad \Longrightarrow \quad \dots s_{i-1} \dots$$

If $s \dot{\approx} t$ is deleted using *Simplify-Eq*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_i \dots \quad \Longrightarrow \quad \dots s_{i-1} \rightarrow_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \dots$$

Case (b): A proof step using a rule $s \rightarrow t$ is in $R_\infty \setminus R_*$. This rule must be deleted during the run.

If $s \rightarrow t$ is deleted using *R-Simplify-Rule*:

$$\dots s_{i-1} \rightarrow_{R_\infty} s_i \dots \quad \Longrightarrow \quad \dots s_{i-1} \rightarrow_{R_\infty} s' \leftarrow_{R_\infty} s_i \dots$$

If $s \rightarrow t$ is deleted using *L-Simplify-Rule*:

$$\dots s_{i-1} \rightarrow_{R_\infty} s_i \dots \quad \Longrightarrow \quad \dots s_{i-1} \rightarrow_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \dots$$

Case (c): A subproof has the form $s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1}$.

If there is no overlap or a non-critical overlap:

$$\dots s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1} \dots \Longrightarrow \dots s_{i-1} \rightarrow^*_{R_*} s' \leftarrow^*_{R_*} s_{i+1} \dots$$

If there is a critical pair that has been added using *Deduce*:

$$\dots s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1} \dots \Longrightarrow \quad \dots s_{i-1} \leftrightarrow_{E_\infty} s_{i+1} \dots$$

In all cases, checking that the replacement subproof is smaller than the replaced subproof is routine. □

Theorem 4.37 *Let $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$ be a fair run and let R_0 and E_* be empty. Then*

(1) every proof in $E_\infty \cup R_\infty$ is equivalent to a rewrite proof in R_*,

(2) R_* is equivalent to E_0, and

(3) R_* is convergent.

Proof. (1) By well-founded induction on $\succ_C$ using the previous lemma.

(2) Clearly $\approx_{E_\infty \cup R_\infty} = \approx_{E_0}$. Since $R_* \subseteq R_\infty$, we get $\approx_{R_*} \subseteq \approx_{E_\infty \cup R_\infty}$. On the other hand, by (1), $\approx_{E_\infty \cup R_\infty} \subseteq \approx_{R_*}$.

(3) Since $\rightarrow_{R_*} \subseteq \succ$, R_* is terminating. By (1), R_* is confluent. ☐

4.7 Unfailing Completion

Classical completion:

Try to transform a set E of equations into an equivalent convergent TRS.

Fail, if an equation can neither be oriented nor deleted.

Unfailing completion (Bachmair, Dershowitz and Plaisted):

If an equation cannot be oriented, we can still use *orientable instances* for rewriting.

Note: If $\succ$ is total on ground terms, then every *ground instance* of an equation is trivial or can be oriented.

Goal: Derive a *ground convergent* set of equations.

Let E be a set of equations, let $\succ$ be a reduction ordering.

We define the relation $\rightarrow_{E^\succ}$ by

$s \rightarrow_{E^\succ} t$ iff there exist $(u \approx v) \in E$ or $(v \approx u) \in E$, $p \in \text{pos}(s)$, and $\sigma : X \rightarrow \mathrm{T}_\Sigma(X)$, such that $s|_p = u\sigma$ and $t = s[v\sigma]_p$ *and* $u\sigma \succ v\sigma$.

Note: $\rightarrow_{E^\succ}$ is terminating by construction.

From now on let $\succ$ be a reduction ordering that is total on ground terms.

E is called ground convergent w. r. t. $\succ$, if for all ground terms s and t with $s \leftrightarrow_E^* t$ there exists a ground term v such that $s \rightarrow_{E^\succ}^* v \leftarrow_{E^\succ}^* t$. (Analogously for $E \cup R$.)

As for standard completion, we establish ground convergence by computing critical pairs.

However, the ordering $\succ$ is not total on non-ground terms. Since $s\theta \succ t\theta$ implies $s \not\preceq t$, we approximate $\succ$ on ground terms by $\not\preceq$ on arbitrary terms.

Let $u_i \dot{\approx} v_i$ $(i = 1, 2)$ be equations in E whose variables have been renamed such that $\text{var}(u_1 \dot{\approx} v_1) \cap \text{var}(u_2 \dot{\approx} v_2) = \emptyset$. Let $p \in \text{pos}(u_1)$ be a position such that $u_1|_p$ is not a variable, σ is an mgu of $u_1|_p$ and u_2, and $u_i\sigma \not\preceq v_i\sigma$ $(i = 1, 2)$. Then $\langle v_1\sigma, (u_1\sigma)[v_2\sigma]_p\rangle$ is called a *semi-critical pair* of E with respect to $\succ$.

The set of all semi-critical pairs of E is denoted by $\text{SP}_\succ(E)$.

Semi-critical pairs of $E \cup R$ are defined analogously. If $\to_R \subseteq \succ$, then $\text{CP}(R)$ and $\text{SP}_\succ(R)$ agree.

Note: In contrast to critical pairs, it may be necessary to consider overlaps of a rule with itself at the top. For instance, if $E = \{f(x) \approx g(y)\}$, then $\langle g(y), g(y')\rangle$ is a non-trivial semi-critical pair.

The *Deduce* rule takes now the following form:

Deduce:

$$\frac{E, \quad R}{E \cup \{s \approx t\}, \quad R} \qquad \text{if } \langle s, t\rangle \in \text{SP}_\succ(E \cup R).$$

Moreover, the fairness criterion for runs is replaced by

$$\text{SP}_\succ(E_* \cup R_*) \subseteq E_\infty$$

(i. e., if every semi-critical pair between persisting rules or equations is computed at some step of the derivation).

Analogously to Thm. 4.37 we obtain now the following theorem:

Theorem 4.38 *Let $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$ be a fair run; let $R_0 = \emptyset$. Then*

(1) $E_* \cup R_*$ is equivalent to E_0, and

(2) $E_* \cup R_*$ is ground convergent.

Moreover one can show that, whenever there exists a *reduced* convergent R such that $\approx_{E_0} = \downarrow_R$ and $\to_R \in \succ$, then for every fair *and simplifying* run $E_* = \emptyset$ and $R_* = R$ up to variable renaming.

Here R is called reduced, if for every $l \to r \in R$, both l and r are irreducible w. r. t. $R \setminus \{l \to r\}$. A run is called simplifying, if R_* is reduced, and for all equations $u \approx v \in E_*$, u and v are incomparable w. r. t. $\succ$ and irreducible w. r. t. R_*.

Unfailing completion is refutationally complete for equational theories:

Theorem 4.39 *Let E be a set of equations, let $\succ$ be a reduction ordering that is total on ground terms. For any two terms s and t, let $\hat{s}$ and $\hat{t}$ be the terms obtained from s and t by replacing all variables by Skolem constants. Let $eq/2$, $true/0$ and $false/0$ be new operator symbols, such that $true$ and $false$ are smaller than all other terms. Let $E_0 = E \cup \{\, eq(\hat{s}, \hat{t}) \approx true,\ eq(x, x) \approx false\}$. If $E_0, \emptyset \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$ be a fair run of unfailing completion, then $s \approx_E t$ iff some $E_i \cup R_i$ contains $true \approx false$.*

Outlook:

Combine ordered resolution and unfailing completion to get a calculus for equational clauses:

compute inferences between (strictly) maximal literals as in ordered resolution, compute overlaps between maximal sides of equations as in unfailing completion

$\Rightarrow$ Superposition calculus.

5 Termination Revisited

So far: Termination as a subordinate task for entailment checking.

TRS is generated by some saturation process; ordering must be chosen before the saturation starts.

Now: Termination as a main task (e. g., for program analysis).

TRS is fixed and known in advance.

Literature:

Nao Hirokawa and Aart Middeldorp: Dependency Pairs Revisited, RTA 2004, pp. 249-268 (in particular Sect. 1–4).

Thomas Arts and Jürgen Giesl: Termination of Term Rewriting Using Dependency Pairs, Theoretical Computer Science, 236:133-178, 2000.

5.1 Dependency Pairs

Invented by T. Arts and J. Giesl in 1996, many refinements since then.

Given: finite TRS R over $\Sigma = (\Omega, \emptyset)$.

$T_0 := \{\, t \in \mathrm{T}_\Sigma(X) \mid$ there is an infinite derivation $t \to_R t_1 \to_R t_2 \to_R \dots \}$.

$T_\infty := \{\, t \in T_0 \mid \forall p > \varepsilon : t|_p \notin T_0 \,\}$ = minimal elements of T_0 w. r. t. $\rhd$.

$t \in T_0 \Rightarrow$ there exists a $t' \in T_\infty$ such that $t \trianglerighteq t'$.

R is non-terminating iff $T_0 \neq \emptyset$ iff $T_\infty \neq \emptyset$.

Assume that $T_\infty \neq \emptyset$ and consider some non-terminating derivation starting from $t \in T_\infty$. Since all subterms of t allow only finite derivations, at some point a rule $l \to r \in R$ must be applied at the root of t (possibly preceded by rewrite steps below the root):

$$t = f(t_1, \dots, t_n) \xrightarrow{>\varepsilon}{}^{*}_{R} f(s_1, \dots, s_n) = l\sigma \xrightarrow{\varepsilon}_{R} r\sigma.$$

In particular, $\mathrm{root}(t) = \mathrm{root}(l)$, so we see that the root symbol of any term in T_∞ must be contained in $D := \{\, \mathrm{root}(l) \mid l \to r \in R \,\}$. D is called the set of *defined symbols* of R; $C := \Omega \setminus D$ is called the set of *constructor symbols* of R.

The term $r\sigma$ is contained in T_0, so there exists a $v \in T_\infty$ such that $r\sigma \trianglerighteq v$.

If v occurred in $r\sigma$ at or below a variable position of r, then $x\sigma|_p = v$ for some $x \in \mathrm{var}(r) \subseteq \mathrm{var}(l)$, hence $s_i \trianglerighteq x\sigma$ and there would be an infinite derivation starting from some t_i. This contradicts $t \in T_\infty$, though.