Resolution Calculus $Res^{\succ}_{\mathrm{sel}}$

The resolution calculus $Res^{\succ}_{\mathrm{sel}}$ is parameterized by

- a selection function sel
- and a well-founded ordering $\succ$ on atoms that is total on ground atoms and stable under substitutions.

(Ground) Ordered Resolution with Selection:

$$\frac{D \vee A \qquad C \vee \neg A}{D \vee C}$$

if the following conditions are satisfied:

(i) $A \succ L$ for all L in D;

(ii) nothing is selected in $D \vee A$ by sel;

(iii) $\neg A$ is selected in $C \vee \neg A$, or nothing is selected in $C \vee \neg A$ and $\neg A \succeq L$ for all L in C.

(Ground) Ordered Factorization with Selection:

$$\frac{C \vee A \vee A}{C \vee A}$$

if the following conditions are satisfied:

(i) $A \succeq L$ for all L in C;

(ii) nothing is selected in $C \vee A \vee A$ by sel.

The extension from ground inferences to non-ground inferences is analogous to ordered resolution (replace $\succ$ by $\not\preceq$ and $\succeq$ by $\not\prec$). Again we assume that $\succ$ is stable under substitutions.

Ordered Resolution with Selection:

$$\frac{D \vee B \qquad C \vee \neg A}{(D \vee C)\sigma}$$

if the following conditions are satisfied:

(i) $\sigma = \mathrm{mgu}(A, B)$;

(ii) $B\sigma \not\preceq L\sigma$ for all L in D;

(iii) nothing is selected in $D \vee B$ by sel;

(iv) $\neg A$ is selected in $C \vee \neg A$, or nothing is selected in $C \vee \neg A$ and $\neg A\sigma \not\prec L\sigma$ for all L in C.

Ordered Factorization with Selection:

$$\frac{C \vee A \vee B}{(C \vee A)\sigma}$$

if the following conditions are satisfied:

(i) $\sigma = \mathrm{mgu}(A, B)$;

(ii) $A\sigma \not\prec L\sigma$ for all L in C.

(iii) nothing is selected in $C \vee A \vee B$ by sel.

Lifting Lemma for $Res^{\succ}_{\text{sel}}$

Lemma 3.40 *Let D and C be variable-disjoint clauses. If*

$$\begin{array}{ccc} D & & C \\ \downarrow \sigma & & \downarrow \rho \\ \underline{D\sigma} & & \underline{C\rho} \\ & C' & \end{array} \qquad \textit{[ground inference in } Res^{\succ}_{\text{sel}}\textit{]}$$

and if $\text{sel}(D\sigma) \simeq \text{sel}(D)$, $\text{sel}(C\rho) \simeq \text{sel}(C)$ *(that is, "corresponding" literals are selected), then there exists a substitution τ such that*

$$\frac{D \qquad C}{C''} \qquad \textit{[inference in } Res^{\succ}_{\text{sel}}\textit{]}$$

$$\begin{array}{c} \downarrow \tau \\ C' = C''\tau \end{array}$$

An analogous lifting lemma holds for factorization.

Saturation of Sets of General Clauses

Corollary 3.41 *Let N be a set of general clauses saturated under $Res^{\succ}_{\text{sel}}$, i. e., $Res^{\succ}_{\text{sel}}(N) \subseteq N$. Then there exists a selection function* sel' *such that* $\text{sel}|_N = \text{sel}'|_N$ *and $G_\Sigma(N)$ is also saturated, i. e.,*

$$Res^{\succ}_{\text{sel}'}(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Proof. We first define the selection function sel' such that $\text{sel}'(C) = \text{sel}(C)$ for all clauses $C \in G_\Sigma(N) \cap N$. For $C \in G_\Sigma(N) \setminus N$ we choose a fixed but arbitrary clause $D \in N$ with $C \in G_\Sigma(D)$ and define $\text{sel}'(C)$ to be those occurrences of literals that are ground instances of the occurrences selected by sel in D. Then proceed as in the proof of Cor. 3.31 using the lifting lemma above. □

Soundness and Refutational Completeness

Theorem 3.42 *Let $\succ$ be an atom ordering and* sel *a selection function such that $Res^{\succ}_{\text{sel}}(N) \subseteq N$. Then*

$$N \models \bot \Leftrightarrow \bot \in N$$

Proof. The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part consider first the propositional level: Construct a candidate interpretation I_N as for unrestricted resolution, except that clauses C in N that have selected literals are not productive, even if they are false in I_C and if their maximal atom occurs only once and is positive. The result for general clauses follows using Corollary 3.41. □

What Do We Gain?

Search spaces become smaller:

1	$P \vee Q$	
2	$P \vee \boxed{\neg Q}$	
3	$\neg P \vee Q$	
4	$\neg P \vee \boxed{\neg Q}$	
5	$Q \vee Q$	Res 1, 3
6	Q	Fact 5
7	$\neg P$	Res 6, 4
8	P	Res 6, 2
9	$\bot$	Res 8, 7

we assume $P \succ Q$ and sel as indicated by $\boxed{X}$. The maximal literal in a clause is depicted in red.

In this example, the ordering and selection function even ensure that the refutation proceeds strictly deterministically.

Rotation redundancy can be avoided:

From

$$\dfrac{\dfrac{C_1 \vee A \quad C_2 \vee \neg A \vee B}{C_1 \vee C_2 \vee B} \qquad C_3 \vee \neg B}{C_1 \vee C_2 \vee C_3}$$

we can obtain by *rotation*

$$\dfrac{C_1 \vee A \qquad \dfrac{C_2 \vee \neg A \vee B \quad C_3 \vee \neg B}{C_2 \vee \neg A \vee C_3}}{C_1 \vee C_2 \vee C_3}$$

another proof of the same clause. In large proofs many rotations are possible. However, if $A \succ B$, then the second proof does not fulfill the ordering restrictions.