Herbrand's Theorem

Lemma 3.33 *Let N be a set of Σ-clauses, let $\mathcal{A}$ be an interpretation. Then $\mathcal{A} \models N$ implies $\mathcal{A} \models G_\Sigma(N)$.*

Lemma 3.34 *Let N be a set of Σ-clauses, let $\mathcal{A}$ be a* Herbrand *interpretation. Then $\mathcal{A} \models G_\Sigma(N)$ implies $\mathcal{A} \models N$.*

Proof. Let $\mathcal{A}$ be a Herbrand model of $G_\Sigma(N)$. We have to show that $\mathcal{A} \models \forall\vec{x}\ C$ for all clauses $\forall\vec{x}\ C$ in N. This is equivalent to $\mathcal{A} \models C$, which in turn is equivalent to $\mathcal{A}(\beta)(C) = 1$ for all assignments β.

Choose $\beta : X \to U_\mathcal{A}$ arbitrarily. Since $\mathcal{A}$ is a Herbrand interpretation, $\beta(x)$ is a ground term for every variable x, so there is a substitution σ such that $x\sigma = \beta(x)$ for all variables x occurring in C. Now let γ be an arbitrary assignment, then for every variable occurring in C we have $(\gamma \circ \sigma)(x) = \mathcal{A}(\gamma)(x\sigma) = x\sigma = \beta(x)$ and consequently $\mathcal{A}(\beta)(C) = \mathcal{A}(\gamma \circ \sigma)(C) = \mathcal{A}(\gamma)(C\sigma)$. Since $C\sigma \in G_\Sigma(N)$ and $\mathcal{A}$ is a Herbrand model of $G_\Sigma(N)$, we get $\mathcal{A}(\gamma)(C\sigma) = 1$, so $\mathcal{A}$ is a model of C. □

Theorem 3.35 (Herbrand) *A set N of Σ-clauses is satisfiable if and only if it has a Herbrand model over Σ.*

Proof. The "⇐" part is trivial. For the "⇒" part let $N \not\models \bot$. Since resolution is sound, this implies that $\bot \notin \mathit{Res}^*(N)$. Obviously, a ground instance of a clause has the same number of literals as the clause itself, so we can conclude that $\bot \notin G_\Sigma(\mathit{Res}^*(N))$. Since $\mathit{Res}^*(N)$ is saturated, $G_\Sigma(\mathit{Res}^*(N))$ is saturated as well by Cor. 3.31. Now $I_{G_\Sigma(\mathit{Res}^*(N))}$ is a Herbrand interpretation over Σ and by Thm. 3.18 it is a model of $G_\Sigma(\mathit{Res}^*(N))$. By Lemma 3.34, every Herbrand model of $G_\Sigma(\mathit{Res}^*(N))$ is a model of $\mathit{Res}^*(N)$. Now $N \subseteq \mathit{Res}^*(N)$, so $I_{G_\Sigma(\mathit{Res}^*(N))} \models N$. □

Corollary 3.36 *A set N of Σ-clauses is satisfiable if and only if its set of ground instances $G_\Sigma(N)$ is satisfiable.*

Proof. The "⇒" part follows directly from Lemma 3.33. For the "⇐" part assume that $G_\Sigma(N)$ is satisfiable. By Thm. 3.35 $G_\Sigma(N)$ has a Herbrand model. By Lemma 3.34, every Herbrand model of $G_\Sigma(N)$ is a model of N. □

Refutational Completeness of General Resolution

Theorem 3.37 *Let N be a set of general clauses that is saturated w. r. t. Res. Then $N \models \bot$ if and only if $\bot \in N$.*

Proof. The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part assume that N is saturated, that is, $Res(N) \subseteq N$. By Corollary 3.31, $G_\Sigma(N)$ is saturated as well, i. e., $Res(G_\Sigma(N)) \subseteq G_\Sigma(N)$. By Cor. 3.36, $N \models \bot$ implies $G_\Sigma(N) \models \bot$. By the refutational completeness of ground resolution, $G_\Sigma(N) \models \bot$ implies $\bot \in G_\Sigma(N)$, so $\bot \in N$. □

3.12 Theoretical Consequences

We get some classical results on properties of first-order logic as easy corollaries.

The Theorem of Löwenheim-Skolem

Theorem 3.38 (Löwenheim–Skolem) *Let Σ be a countable signature and let S be a set of closed Σ-formulas. Then S is satisfiable iff S has a model over a countable universe.*

Proof. If both X and Σ are countable, then S can be at most countably infinite. Now generate, maintaining satisfiability, a set N of clauses from S. This extends Σ by at most countably many new Skolem functions to Σ'. As Σ' is countable, so is $T_{\Sigma'}$, the universe of Herbrand-interpretations over Σ'. Now apply Theorem 3.35. □

There exist more refined versions of this theorem. For instance, one can show that, if S has some infinite model, then S has a model with a universe of cardinality κ for every κ that is larger than or equal to the cardinalty of the signature Σ.

Compactness of Predicate Logic

Theorem 3.39 (Compactness Theorem for First-Order Logic) *Let S be a set of closed first-order formulas. S is unsatisfiable $\Leftrightarrow$ some finite subset $S' \subseteq S$ is unsatisfiable.*

Proof. The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let S be unsatisfiable and let N be the set of clauses obtained by Skolemization and CNF transformation of the formulas in S. Clearly $Res^*(N)$ is unsatisfiable. By Theorem 3.37, $\bot \in Res^*(N)$, and therefore $\bot \in Res^n(N)$ for some $n \in \mathbb{N}$. Consequently, $\bot$ has a finite resolution proof B of depth $\leq n$. Choose S' as the subset of formulas in S such that the corresponding clauses contain the assumptions (leaves) of B. □

3.13 Ordered Resolution with Selection

Motivation: Search space for *Res* *very* large.

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem 3.18) one only needs to resolve and factor maximal atoms
   $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
   $\Rightarrow$ *ordering restrictions*

2. In the proof, it does not really matter with which negative literal an inference is performed
   $\Rightarrow$ choose a negative literal don't-care-nondeterministically
   $\Rightarrow$ *selection*

Ordering Restrictions

In the completeness proof one only needs to resolve and factor maximal atoms
$\Rightarrow$ If we impose ordering restrictions on ground inferences, the proof remains correct:

(Ground) Ordered Resolution:

$$\frac{D \vee A \qquad C \vee \neg A}{D \vee C}$$

if $A \succ L$ for all L in D and $\neg A \succeq L$ for all L in C.

(Ground) Ordered Factorization:

$$\frac{C \vee A \vee A}{C \vee A}$$

if $A \succeq L$ for all L in C.

Problem: How to extend this to non-ground inferences?

In the completeness proof, we talk about (strictly) maximal literals of *ground* clauses.

In the non-ground calculus, we have to consider those literals that correspond to (strictly) maximal literals of ground instances.

An ordering $\succ$ on atoms (or terms) is called *stable under substitutions*, if $A \succ B$ implies $A\sigma \succ B\sigma$.

Note:

- We can not require that $A \succ B$ iff $A\sigma \succ B\sigma$.
- We can not require that $\succ$ is total on non-ground atoms.

Consequence: In the ordering restrictions for non-ground inferences, we have to replace $\succ$ by $\not\preceq$ and $\succeq$ by $\not\prec$.

Ordered Resolution:

$$\frac{D \vee B \qquad C \vee \neg A}{(D \vee C)\sigma}$$

if $\sigma = \mathrm{mgu}(A, B)$ and $B\sigma \not\preceq L\sigma$ for all L in D and $\neg A\sigma \not\prec L\sigma$ for all L in C.

Ordered Factorization:

$$\frac{C \vee A \vee B}{(C \vee A)\sigma}$$

if $\sigma = \mathrm{mgu}(A, B)$ and $A\sigma \not\prec L\sigma$ for all L in C.

Selection Functions

Selection functions can be used to override ordering restrictions for individual clauses.

A *selection function* is a mapping

$$\mathrm{sel} : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \vee \neg A \vee B$$

$$\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$$

Intuition:

- If a clause has at least one selected literal, compute only inferences that involve a selected literal.
- If a clause has no selected literals, compute only inferences that involve a maximal literal.