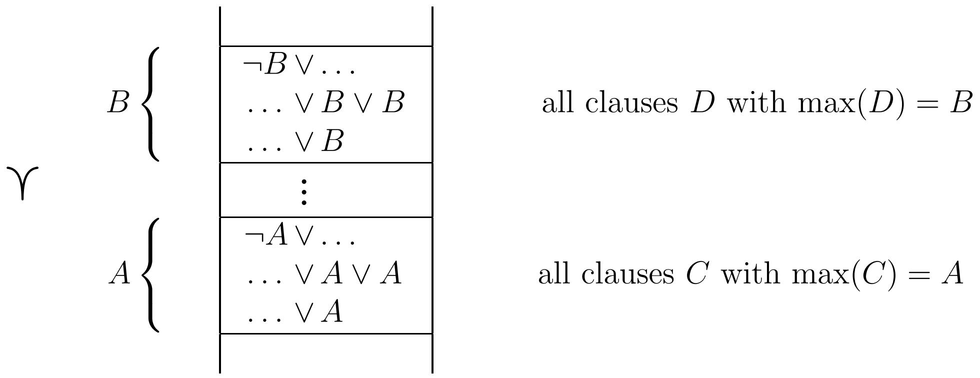
Properties of the Clause Ordering

Proposition 3.14

1. *The orderings on literals and clauses are total and well-founded.*
2. *Let C and D be clauses with $A = \max(C)$, $B = \max(D)$, where $\max(C)$ denotes the maximal atom in C.*
   (i) If $A \succ B$ then $C \succ D$.
   (ii) If $A = B$, A occurs negatively in C but only positively in D, then $C \succ D$.

Stratified Structure of Clause Sets

Let $B \succ A$. Clause sets are then stratified in this form:



Closure of Clause Sets under *Res*

$$
\begin{aligned}
Res(N) &= \{ C \mid C \text{ is conclusion of an inference in } Res \\
&\qquad\qquad \text{with premises in } N \} \\
Res^0(N) &= N \\
Res^{n+1}(N) &= Res(Res^n(N)) \cup Res^n(N), \text{ for } n \geq 0 \\
Res^*(N) &= \textstyle\bigcup_{n\geq 0} Res^n(N)
\end{aligned}
$$

N is called *saturated* (w. r. t. resolution), if $Res(N) \subseteq N$.

Proposition 3.15

(i) $Res^*(N)$ is saturated.

(ii) Res is refutationally complete, iff for each set N of ground clauses:

$$N \models \bot \;\; \textit{implies} \;\; \bot \in Res^*(N)$$

Construction of Interpretations

Given: set N of ground clauses, atom ordering $\succ$.
Wanted: Herbrand interpretation I such that

- "many" clauses from N are valid in I;
- *$I \models N$, if N is saturated and $\bot \notin N$.*

Construction according to $\succ$, starting with the smallest clause.

Main Ideas of the Construction

- Clauses are considered in the order given by $\succ$.
- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.
- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).
- If C is false, one would like to change I_C such that C becomes true.
- Changes should, however, be *monotone*. One never deletes anything from I_C and the truth value of clauses smaller than C should remain as it was in I_C.
- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, C is false in I_C, if A occurs positively in C (*adding A will make C become true*) and if this occurrence in C is strictly maximal in the ordering on literals (*changing the truth value of A has no effect on smaller clauses*).
- We say that the construction fails for a clause C, if C is false in I_C and $\Delta_C = \emptyset$.
- We will show: If there are clauses for which the construction fails, then some inference with the smallest such clause (the so-called "minimal counterexample") has not been computed. Otherwise, the limit interpretation is a model of all clauses.

Construction of Candidate Interpretations

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

We say that C *produces* A, if $\Delta_C = \{A\}$.

Note that the definitions satisfy the conditions of Thm. 1.8; so they are well-defined even if $\{ D \mid C \succ D \}$ is infinite.

The *candidate interpretation* for N (w. r. t. $\succ$) is given as $I_N^{\succ} := \bigcup_C \Delta_C$. (We also simply write I_N or I for $I_N^{\succ}$ if $\succ$ is either irrelevant or known from the context.)

Example

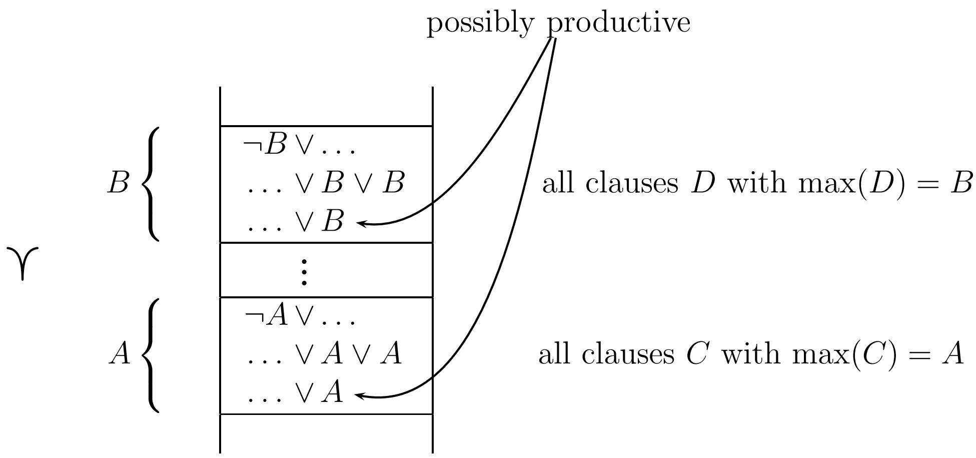
Let $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$ (max. literals in red)

	clauses C	I_C	Δ_C	Remarks
7	$\neg A_1 \vee A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	
6	$\neg A_1 \vee A_3 \vee \neg A_4$	$\{A_1, A_2, A_4\}$	$\emptyset$	max. lit. $\neg A_4$ neg.; *min. counter-ex.*
5	$A_0 \vee \neg A_1 \vee A_3 \vee A_4$	$\{A_1, A_2\}$	$\{A_4\}$	A_4 maximal
4	$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	A_2 maximal
3	$A_1 \vee A_2$	$\{A_1\}$	$\emptyset$	true in I_C
2	$A_0 \vee A_1$	$\emptyset$	$\{A_1\}$	A_1 maximal
1	$\neg A_0$	$\emptyset$	$\emptyset$	true in I_C

$I = \{A_1, A_2, A_4, A_5\}$ is not a model of the clause set
$\Rightarrow$ there exists a *counterexample*.

Structure of $N, \succ$

Let $B \succ A$. Note that producing a new atom does change the truth value of smaller clauses.



Some Properties of the Construction

Proposition 3.16

(i) *If* $D = D' \vee \neg A$, *then no* $C \succeq D$ *produces* A.

(ii) *If* $I_D \models D$, *then* $I_C \models D$ *for every* $C \succeq D$ *and* $I_N^{\succ} \models D$.

(iii) *If* $D = D' \vee A$ *produces* A, *then* $I_C \models D$ *for every* $C \succ D$ *and* $I_N^{\succ} \models D$.

(iv) *If* $D = D' \vee A$ *produces* A, *then* $I_C \not\models D'$ *for every* $C \succeq D$ *and* $I_N^{\succ} \not\models D'$.

(v) *If for every clause* $C \in N$, C *is productive or* $I_C \models C$, *then* $I_N^{\succ} \models N$.

Proof. *(i) If* C *produces* A, *then* $A \succeq L$ *for every literal* L *of* C. *On the other hand,* D *contains* $\neg A$, *and* $\neg A \succ A$. *Since* $\neg A \succ L$ *for every literal* L *of* C, *we obtain* $D \succ C$.

(ii) Suppose that $I_D \models D$ *and* $C \succeq D$. *If* $I_D \models A$ *for some positive literal* A *of* D, *then* $A \in I_D \subseteq I_C \subseteq I_N^{\succ}$, *so* $I_C \models D$ *and* $I_N^{\succ} \models D$. *Otherwise* $I_D \models \neg A$ *for some negative literal* $\neg A$ *of* D, *hence* $A \notin I_D$. *By (i), no clause that is larger than or equal to* D *produces* A, *so* $A \notin I_C$ *and* $A \notin I_N^{\succ}$. *Again,* $I_C \models D$ *and* $I_N^{\succ} \models D$.

(iii) Obvious, since $C \succ D$ *implies* $A \in \Delta_D \subseteq I_C \subseteq I_N^{\succ}$.

(iv) If $D = D' \vee A$ *produces* A, *then* $A \succ L$ *for every literal* L *of* D' *and* $I_D \not\models A$. *Hence* $I_D \not\models L$ *for every literal* L *of* D'. *Let* $C \succeq D$. *If* L *is a positive literal* A', *then* $A' \notin I_D$. *Since all atoms in* $I_C \setminus I_D$ *and* $I_N^{\succ} \setminus I_D$ *are larger than or equal to* A, *we get* $A' \notin I_C$ *and* $A' \notin I_N^{\succ}$. *Otherwise* L *is a negative literal* $\neg A'$, *then obviously* $A' \in I_D \subseteq I_C \subseteq I_N^{\succ}$. *In both cases* L *is false in* I_C *and* $I_N^{\succ}$.

(v) By (ii) and (iii). □

Model Existence Theorem

Proposition 3.17 *Let* $\succ$ *be a clause ordering. If* N *is saturated* w. r. t. *Res and* $\bot \notin N$, *then for every clause* $C \in N$, C *is productive or* $I_C \models C$.

Proof. Let N be saturated w. r. t. *Res* and $\bot \notin N$. Assume that the proposition does not hold. By well-foundedness, there must exist a minimal clause $C \in N$ (w. r. t. $\succ$) such that C is neither productive nor $I_C \models C$. As $C \neq \bot$ there exists a maximal literal in C. There are two possible reasons why C is not productive:

Case 1: The maximal literal $\neg A$ is negative, i. e., $C = C' \vee \neg A$. Then $I_C \models A$ and $I_C \not\models C'$. So some $D = D' \vee A \in N$ with $C \succ D$ produces A, and $I_C \not\models D'$. The inference

$$\frac{D' \vee A \qquad C' \vee \neg A}{D' \vee C'}$$

yields a clause $D' \vee C' \in N$ that is smaller than C. As $I_C \not\models D' \vee C'$, we know that $D' \vee C'$ is neither productive nor $I_{D' \vee C'} \models D' \vee C'$. This contradicts the minimality of C.

Case 2: The maximal literal A is positive, but not strictly maximal, i. e., $C = C' \vee A \vee A$. Then there is an inference

$$\frac{C' \vee A \vee A}{C' \vee A}$$

that yields a smaller clause $C' \vee A \in N$. As $I_C \not\models C' \vee A$, this clause is neither productive nor $I_{C' \vee A} \models C' \vee A$. Since $C \succ C' \vee A$, this contradicts the minimality of C. □

Theorem 3.18 (Bachmair & Ganzinger 1990) *Let $\succ$ be a clause ordering. If N is saturated w. r. t. Res and $\bot \notin N$, then $I_N^{\succ} \models N$.*

Proof. By Prop. 3.17 and part (v) of Prop. 3.16. □

Corollary 3.19 *Let N be saturated w. r. t. Res. Then $N \models \bot$ if and only if $\bot \in N$.*

Compactness of Propositional Logic

Lemma 3.20 *Let N be a set of propositional (or first-order ground) clauses. Then N is unsatisfiable, if and only if some finite subset $N' \subseteq N$ is unsatisfiable.*

Proof. The "if" part is trivial. For the "only if" part, assume that N be unsatisfiable. Consequently, $Res^*(N)$ unsatisfiable as well. By refutational completeness of resolution, $\bot \in Res^*(N)$. So there exists an $n \geq 0$ such that $\bot \in Res^n(N)$, which means that $\bot$ has a finite resolution proof. Now choose N' as the set of assumptions in this proof. □

Theorem 3.21 (Compactness for Propositional Formulas) *Let S be a set of propositional (or first-order ground) formulas. Then S is unsatisfiable, if and only if some finite subset $S' \subseteq S$ is unsatisfiable.*

Proof. The "if" part is again trivial. For the "only if" part, assume that S be unsatisfiable. Transform S into an equivalent set N of clauses. By the previous lemma, N has a finite unsatisfiable subset N'. Now choose for every clause C in N' one formula F of S such that C is contained in the CNF of F. Let S' be the set of these formulas. □

3.11 General Resolution

Propositional (ground) resolution:

refutationally complete,

in its most naive version: not guaranteed to terminate for satisfiable sets of clauses, (improved versions do terminate, however)

inferior to the CDCL procedure.

But: in contrast to the CDCL procedure, resolution can be easily extended to non-ground clauses.

Two Lemmas

Lemma 3.22 *Let $\mathcal{A}$ be a Σ-algebra and let F be a Σ-formula with free variables $x_1, \ldots, x_n$. Then*

$$\mathcal{A} \models \forall x_1, \ldots, x_n\, F \quad \text{if and only if} \quad \mathcal{A} \models F$$

Lemma 3.23 *Let F be a Σ-formula with free variables $x_1, \ldots, x_n$, let σ be a substitution, and let $y_1, \ldots, y_m$ be the free variables of $F\sigma$. Then*

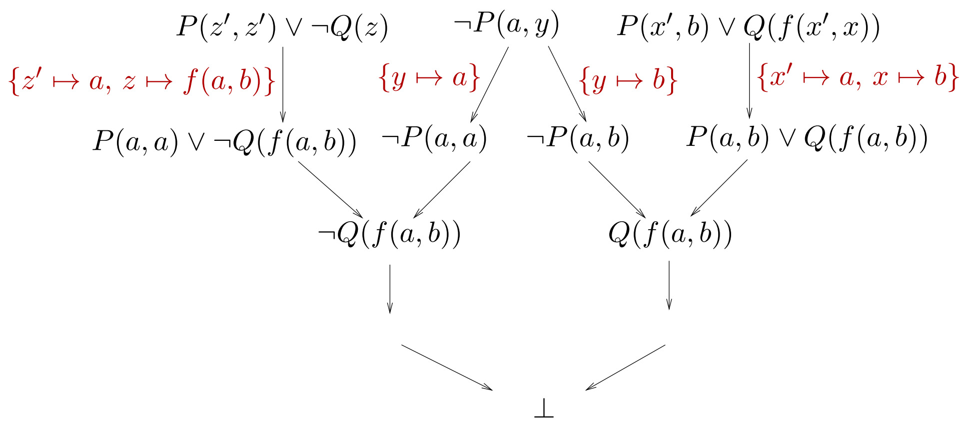
$$\mathcal{A} \models \forall x_1, \ldots, x_n\, F \quad \text{implies} \quad \mathcal{A} \models \forall y_1, \ldots, y_m\, F\sigma$$

In particular, if $\mathcal{A}$ is a model of an (implicitly universally quantified) clause C, then it is also a model of all (implicitly universally quantified) instances $C\sigma$ of C.

Consequently, if we show that some instances of clauses in a set N are unsatisfiable, then we have also shown that N itself is unsatisfiable.

General Resolution through Instantiation

Idea: instantiate clauses appropriately:



Early approaches (Gilmore 1960, Davis and Putnam 1960):

Generate ground instances of clauses.

Try to refute the set of ground instances by resolution.

If no contradiction is found, generate more ground instances.

Problems:

More than one instance of a clause can participate in a proof.

Even worse: There are infinitely many possible instances.

Observation:

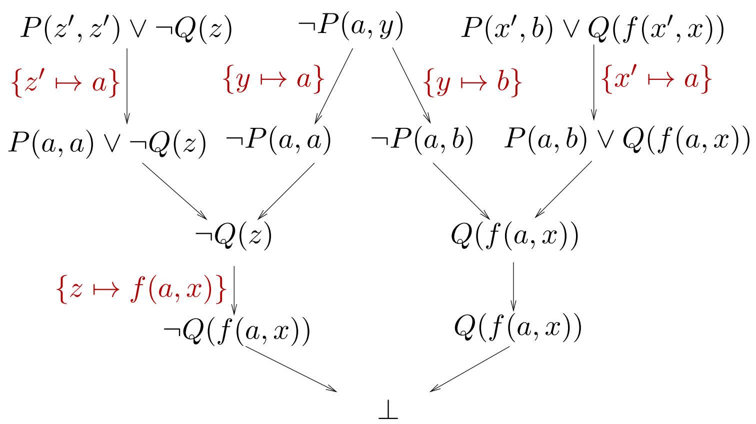
Instantiation must produce complementary literals (so that inferences become possible).

Idea (Robinson 1965):

Do not instantiate more than necessary to get complementary literals
⇒ most general unifiers (mgu).

Calculus works with non-ground clauses; inferences with non-ground clauses represent infinite sets of ground inferences which are computed simultaneously
⇒ lifting principle.

Computation of instances becomes a by-product of boolean reasoning.



Unification

Let $E = \{s_1 \doteq t_1, \ldots, s_n \doteq t_n\}$ (s_i, t_i terms or atoms) be a multiset of *equality problems.* A substitution σ is called a *unifier* of E if $s_i\sigma = t_i\sigma$ for all $1 \leq i \leq n$.

If a unifier of E exists, then E is called *unifiable.*

A substitution σ is called *more general* than a substitution τ, denoted by $\sigma \leq \tau$, if there exists a substitution ρ such that $\rho \circ \sigma = \tau$, where $(\rho \circ \sigma)(x) := (x\sigma)\rho$ is the composition of σ and ρ as mappings. (Note that $\rho \circ \sigma$ has a finite domain as required for a substitution.)

If a unifier of E is more general than any other unifier of E, then we speak of a *most general unifier* of E, denoted by $\mathrm{mgu}(E)$.

Proposition 3.24

(i) $\leq$ *is a quasi-ordering on substitutions, and* $\circ$ *is associative.*

(ii) If $\sigma \leq \tau$ *and* $\tau \leq \sigma$ *(we write* $\sigma \sim \tau$ *in this case), then* $x\sigma$ *and* $x\tau$ *are equal up to (bijective) variable renaming, for any* x *in* X.

A substitution σ is called *idempotent,* if $\sigma \circ \sigma = \sigma$.

Proposition 3.25 *σ is idempotent iff* $\mathrm{dom}(\sigma) \cap \mathrm{codom}(\sigma) = \emptyset$.

Rule-Based Naive Standard Unification

$$
\begin{array}{rcl}
t \doteq t, E & \Rightarrow_{SU} & E \\
f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{SU} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) \doteq g(\ldots), E & \Rightarrow_{SU} & \bot \\
 & & \text{if } f \neq g \\
x \doteq t, E & \Rightarrow_{SU} & x \doteq t, E\{x \mapsto t\} \\
 & & \text{if } x \in \mathrm{var}(E), x \notin \mathrm{var}(t) \\
x \doteq t, E & \Rightarrow_{SU} & \bot \\
 & & \text{if } x \neq t, x \in \mathrm{var}(t) \\
t \doteq x, E & \Rightarrow_{SU} & x \doteq t, E \\
 & & \text{if } t \notin X
\end{array}
$$