max planck institut
informatik

Universität
des
Saarlandes
FR Informatik

Uwe Waldmann

October 26, 2017

Tutorials for "Automated Reasoning"
Exercise sheet 1

Exercise 1.1: (*4 P*)
Find an abstract reduction system $(A, \to)$, such that $\to^+$ is irreflexive and $\to$ is normalizing, but not terminating.

Exercise 1.2: (*4 P*)
For an alphabet Σ with a well-founded ordering $>_\Sigma$ let the relation $>_{\Sigma,lex} \subseteq \Sigma^* \times \Sigma^*$ be defined by $w >_{\Sigma,lex} w'$ if and only if w and w' have the same length n and w is larger than w' in the n-fold lexicographic combination of $>_\Sigma$. Let the relation $\succ$ be defined as

$$w \succ w' :\Leftrightarrow |w| > |w'| \text{ or } (|w| = |w'| \text{ and } w >_{\Sigma,lex} w').$$

Prove that $\succ$ is a well-founded ordering on Σ^*. (Note: We define the 0-fold lexicographic combination of an ordering as $\emptyset$ and the 1-fold lexicographic combination of an ordering as the ordering itself. You may use the fact that for any $n \geq 1$ the n-fold lexicographic combination of a well-founded ordering is well-founded.)

Exercise 1.3: (*4 P*)
Let M be the set $\{a, b, c\}$ and let the ordering $\succ$ be defined by $a \succ b$ and $a \succ c$. (b and c are incomparable!) Consider the following multisets over multisets over M:

(1) $\big\{\{a, a\}\big\}$

(2) $\big\{\{a, b, c\}, \{b, c\}\big\}$

(3) $\big\{\{a, b, c\}, \{b, b\}, \{c, c\}\big\}$

(4) $\big\{\{a, c\}, \{b, b, c, c\}\big\}$

Determine for each pair of multisets whether they are comparable with respect to $(\succ_{\text{mul}})_{\text{mul}}$, and, if so, which multiset is larger.

Exercise 1.4: (*6 P*)
You are asked to review a scientific article that has been submitted to a conference on automated reasoning. On page 3 of the article, the authors write the following:

> **Theorem 2.** Let $\rightarrow_1$ and $\rightarrow_2$ be two binary relations over a non-empty set M. If $\rightarrow_1$ and $\rightarrow_2$ are terminating, then $\rightarrow_1 \cup \rightarrow_2$ is also terminating.
>
> **Proof.** Since $\rightarrow_1$ is terminating, $\rightarrow_1^+$ is a well-founded ordering. Assume that there exists an infinite $(\rightarrow_1 \cup \rightarrow_2)$-derivation. Since $\rightarrow_1^+$ is well-founded, there exists a minimal element b with respect to $\rightarrow_1^+$ such that there is an infinite $(\rightarrow_1 \cup \rightarrow_2)$-derivation starting with b.
>
> Case 1: The $(\rightarrow_1 \cup \rightarrow_2)$-derivation starts with a $\rightarrow_1$-step $b \rightarrow_1 b'$. The rest of the derivation, starting with b', is still infinite. However, b' is smaller than b with respect to $\rightarrow_1^+$. This contradicts the minimality of b.
>
> Case 2: The $(\rightarrow_1 \cup \rightarrow_2)$-derivation starts with a $\rightarrow_2$-step. Since $\rightarrow_2$ is terminating, the derivation cannot consist only of $\rightarrow_2$-steps. Let $b' \rightarrow_1 b''$ be the first $\rightarrow_1$-step in the derivation, then there exists an infinite $(\rightarrow_1 \cup \rightarrow_2)$-derivation starting with a $\rightarrow_1$-step. But as we have seen in Case 1, an infinite $(\rightarrow_1 \cup \rightarrow_2)$-derivation cannot start with a $\rightarrow_1$-step. So there is again a contradiction.
>
> Consequently, every $(\rightarrow_1 \cup \rightarrow_2)$-derivation must be finite, which means that $\rightarrow_1 \cup \rightarrow_2$ is terminating.

(1) Is the “proof” correct (yes/no)?

(2) If the “proof” is not correct:

  (a) Which step is incorrect?

  (b) Does the “theorem” hold? If yes, give a correct proof, otherwise give a counterexample.

Submit your solution in lecture hall E1.3, Room 003 during the lecture on November 2. Please write your name and the time of your tutorial group (Tue 8–10, Wed 8–10, or Wed 14–16) on your solution.

Joint solutions, prepared by up to three persons together, are allowed (but not encouraged). If you prepare your solution jointly, submit it only once and indicate all authors on the sheet.