3.6 Getting Skolem Functions with Small Arity

A clause set that is better suited for automated theorem proving can be obtained using the following steps:

- eliminate trivial subformulas
- replace beneficial subformulas
- produce a negation normal form (NNF)
- apply miniscoping
- rename all variables
- Skolemize
- push quantifiers upward
- apply distributivity

We start with a closed formula.

Elimination of Trivial Subformulas

Eliminate subformulas $\top$ and $\bot$ essentially as in the propositional case modulo associativity/commutativity of $\wedge$, $\vee$:

$$\begin{aligned}
H[(F \wedge \top)]_p &\Rightarrow_{\text{OCNF}} H[F]_p \\
H[(F \vee \bot)]_p &\Rightarrow_{\text{OCNF}} H[F]_p \\
H[(F \leftrightarrow \bot)]_p &\Rightarrow_{\text{OCNF}} H[\neg F]_p \\
H[(F \leftrightarrow \top)]_p &\Rightarrow_{\text{OCNF}} H[F]_p \\
H[(F \vee \top)]_p &\Rightarrow_{\text{OCNF}} H[\top]_p \\
H[(F \wedge \bot)]_p &\Rightarrow_{\text{OCNF}} H[\bot]_p \\
H[\neg\top]_p &\Rightarrow_{\text{OCNF}} H[\bot]_p \\
H[\neg\bot]_p &\Rightarrow_{\text{OCNF}} H[\top]_p \\
H[(F \rightarrow \bot)]_p &\Rightarrow_{\text{OCNF}} H[\neg F]_p \\
H[(F \rightarrow \top)]_p &\Rightarrow_{\text{OCNF}} H[\top]_p \\
H[(\bot \rightarrow F)]_p &\Rightarrow_{\text{OCNF}} H[\top]_p \\
H[(\top \rightarrow F)]_p &\Rightarrow_{\text{OCNF}} H[F]_p \\
H[Qx\,\top]_p &\Rightarrow_{\text{OCNF}} H[\top]_p \\
H[Qx\,\bot]_p &\Rightarrow_{\text{OCNF}} H[\bot]_p
\end{aligned}$$

Replacement of Beneficial Subformulas

The functions ν and $\bar{\nu}$ that give us an overapproximation for the number of clauses generated by a formula are extended to quantified formulas by

$$\begin{aligned}&\nu(\forall x\, F) = \nu(\exists x\, F) = \nu(F),\\ &\bar{\nu}(\forall x\, F) = \bar{\nu}(\exists x\, F) = \bar{\nu}(F).\end{aligned}$$

The other cases are defined as for propositional formulas.

Introduce top-down fresh predicates for beneficial subformulas:

$$H[F]_p \ \Rightarrow_{\mathrm{OCNF}}\ H[P(x_1,\dots,x_n)]_p \wedge \mathrm{def}(H,p,P,F)$$

if $\nu(H[F]_p) > \nu(H[P(\dots)]_p \wedge \mathrm{def}(H,p,P,F))$,

where $\{x_1,\dots,x_n\}$ are the free variables in F, P/n is a predicate new to $H[F]_p$, and $\mathrm{def}(H,p,P,F)$ is defined by

$$\begin{aligned}&\forall x_1,\dots,x_n\ (P(x_1,\dots,x_n) \rightarrow F),\ \text{if}\ \mathrm{pol}(H,p)=1,\\ &\forall x_1,\dots,x_n\ (F \rightarrow P(x_1,\dots,x_n)),\ \text{if}\ \mathrm{pol}(H,p)=-1,\\ &\forall x_1,\dots,x_n\ (P(x_1,\dots,x_n) \leftrightarrow F),\ \text{if}\ \mathrm{pol}(H,p)=0.\end{aligned}$$

As in the propositional case, one can test $\nu(H[F]_p) > \nu(H[P]_p \wedge \mathrm{def}(H,p,P,F))$ in constant time without actually computing ν.

Negation Normal Form (NNF)

Apply the reduction system $\Rightarrow_{\mathrm{NNF}}$:

$$H[F \leftrightarrow G]_p \ \Rightarrow_{\mathrm{NNF}}\ H[(F \rightarrow G) \wedge (G \rightarrow F)]_p$$

if $\mathrm{pol}(H,p)=1$ or $\mathrm{pol}(H,p)=0$.

$$H[F \leftrightarrow G]_p \ \Rightarrow_{\mathrm{NNF}}\ H[(F \wedge G) \vee (\neg G \wedge \neg F)]_p$$

if $\mathrm{pol}(H,p)=-1$.

$$\begin{aligned}H[F \rightarrow G]_p &\ \Rightarrow_{\mathrm{NNF}}\ H[\neg F \vee G]_p\\ H[\neg\neg F]_p &\ \Rightarrow_{\mathrm{NNF}}\ H[F]_p\\ H[\neg(F \vee G)]_p &\ \Rightarrow_{\mathrm{NNF}}\ H[\neg F \wedge \neg G]_p\\ H[\neg(F \wedge G)]_p &\ \Rightarrow_{\mathrm{NNF}}\ H[\neg F \vee \neg G]_p\\ H[\neg Qx\ F]_p &\ \Rightarrow_{\mathrm{NNF}}\ H[\overline{Q}x\ \neg F]_p\end{aligned}$$

Miniscoping

Apply the reduction system $\Rightarrow_{\mathrm{MS}}$ modulo associativity and commutativity of $\wedge$, $\vee$. For the rules below we assume that x occurs freely in F, F', but x does not occur freely in G:

$$\begin{aligned}
H[Qx\,(F \wedge G)]_p &\Rightarrow_{\mathrm{MS}} H[(Qx\,F) \wedge G]_p\\
H[Qx\,(F \vee G)]_p &\Rightarrow_{\mathrm{MS}} H[(Qx\,F) \vee G]_p\\
H[\forall x\,(F \wedge F')]_p &\Rightarrow_{\mathrm{MS}} H[(\forall x\,F) \wedge (\forall x\,F')]_p\\
H[\exists x\,(F \vee F')]_p &\Rightarrow_{\mathrm{MS}} H[(\exists x\,F) \vee (\exists x\,F')]_p\\
H[Qx\,G]_p &\Rightarrow_{\mathrm{MS}} H[G]_p
\end{aligned}$$

Variable Renaming

Rename all variables in H such that there are no two different positions p, q with $H|_p = Qx\,F$ and $H|_q = Q'x\,G$.

Standard Skolemization

Apply the reduction system:

$$H[\exists x\,F]_p \Rightarrow_{\mathrm{SK}} H[F\{x \mapsto f(y_1, \ldots, y_n)\}]_p$$

where p has minimal length,
$\{y_1, \ldots, y_n\}$ are the free variables in $\exists x\,F$,
and f/n is a new function symbol to H.

Final Steps

Apply the reduction system modulo commutativity of $\wedge$, $\vee$ to push $\forall$ upward:

$$\begin{aligned}
H[(\forall x\,F) \wedge G]_p &\Rightarrow_{\mathrm{OCNF}} H[\forall x\,(F \wedge G)]_p\\
H[(\forall x\,F) \vee G]_p &\Rightarrow_{\mathrm{OCNF}} H[\forall x\,(F \vee G)]_p
\end{aligned}$$

Note that variable renaming ensures that x does not occur in G.

Apply the reduction system modulo commutativity of $\wedge$, $\vee$ to push disjunctions downward:

$$H[(F \wedge F') \vee G]_p \Rightarrow_{\mathrm{CNF}} H[(F \vee G) \wedge (F' \vee G)]_p$$

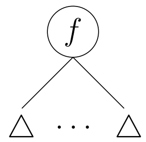
3.7 Herbrand Interpretations

From now on we shall consider FOL without equality. We assume that Ω contains at least one constant symbol.

A *Herbrand interpretation* (over Σ) is a Σ-algebra $\mathcal{A}$ such that

- $U_{\mathcal{A}} = \mathrm{T}_{\Sigma}$ (= the set of ground terms over Σ)
- $f_{\mathcal{A}} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n),\ f/n \in \Omega$

$$f_{\mathcal{A}}(\triangle, \ldots, \triangle) =$$



In other words, *values are fixed* to be ground terms and *functions are fixed* to be the *term constructors*. Only predicate symbols $P/m \in \Pi$ may be freely interpreted as relations $P_{\mathcal{A}} \subseteq \mathrm{T}_{\Sigma}^m$.

Proposition 3.10 *Every set of ground atoms I uniquely determines a Herbrand interpretation $\mathcal{A}$ via*

$$(s_1, \ldots, s_n) \in P_{\mathcal{A}} \ \text{ iff } \ P(s_1, \ldots, s_n) \in I$$

Thus we shall identify Herbrand interpretations (over Σ) with sets of Σ-ground atoms.

Existence of Herbrand Models

A Herbrand interpretation I is called a *Herbrand model* of F, if $I \models F$.

Theorem 3.11 (Herbrand) *Let N be a set of (universally quantified) Σ-clauses.*

$$\begin{array}{lcl} N \text{ satisfiable} & \Leftrightarrow & N \text{ has a Herbrand model (over } \Sigma) \\ & \Leftrightarrow & G_{\Sigma}(N) \text{ has a Herbrand model (over } \Sigma) \end{array}$$

where $G_{\Sigma}(N) = \{\, C\sigma \text{ ground clause} \mid (\forall \vec{x}\, C) \in N,\ \sigma : X \to \mathrm{T}_{\Sigma} \,\}$ *is the set of* ground instances *of N.*

[The proof will be given below in the context of the completeness proof for general resolution.]