Saturation of Sets of General Clauses

Corollary 3.30 *Let N be a set of general clauses saturated under Res, i. e., $Res(N) \subseteq N$. Then also $G_\Sigma(N)$ is saturated, that is,*

$$Res(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Proof. W.l.o.g. we may assume that clauses in N are pairwise variable-disjoint. (Otherwise make them disjoint, and this renaming process changes neither $Res(N)$ nor $G_\Sigma(N)$.)

Let $C' \in Res(G_\Sigma(N))$, meaning (i) there exist resolvable ground instances $D\sigma$ and $C\rho$ of N with resolvent C', or else (ii) C' is a factor of a ground instance $C\sigma$ of C.

Case (i): By the Lifting Lemma, D and C are resolvable with a resolvent C'' with $C''\tau = C'$, for a suitable substitution τ. As $C'' \in N$ by assumption, we obtain that $C' \in G_\Sigma(N)$.

Case (ii): Similar. $\square$

Soundness for General Clauses

Proposition 3.31 *The general resolution calculus is sound.*

Proof. We have to show that, if $\sigma = \mathrm{mgu}(A, B)$ then $\{\forall\vec{x}\ (D \vee B), \forall\vec{y}\ (C \vee \neg A)\} \models \forall\vec{z}\ (D \vee C)\sigma$ and $\{\forall\vec{x}\ (C \vee A \vee B)\} \models \forall\vec{z}\ (C \vee A)\sigma$.

Let $\mathcal{A}$ be a model of $\forall\vec{x}\ (D \vee B)$ and $\forall\vec{y}\ (C \vee \neg A)$. By Lemma 3.22, $\mathcal{A}$ is also a model of $\forall\vec{z}\ (D \vee B)\sigma$ and $\forall\vec{z}\ (C \vee \neg A)\sigma$ and by Lemma 3.21, $\mathcal{A}$ is also a model of $(D \vee B)\sigma$ and $(C \vee \neg A)\sigma$. Let β be an assignment. If $\mathcal{A}(\beta)(B\sigma) = 0$, then $\mathcal{A}(\beta)(D\sigma) = 1$. Otherwise $\mathcal{A}(\beta)(B\sigma) = \mathcal{A}(\beta)(A\sigma) = 1$, hence $\mathcal{A}(\beta)(\neg A\sigma) = 0$ and therefore $\mathcal{A}(\beta)(C\sigma) = 1$. In both cases $\mathcal{A}(\beta)((D \vee C)\sigma) = 1$, so $\mathcal{A} \models (D \vee C)\sigma$ and by Lemma 3.21, $\mathcal{A} \models \forall\vec{z}\ (D \vee C)\sigma$.

The proof for factorization inferences is similar. $\square$

Herbrand's Theorem

Lemma 3.32 *Let N be a set of Σ-clauses, let $\mathcal{A}$ be an interpretation. Then $\mathcal{A} \models N$ implies $\mathcal{A} \models G_\Sigma(N)$.*

Lemma 3.33 *Let N be a set of Σ-clauses, let $\mathcal{A}$ be a* Herbrand *interpretation. Then $\mathcal{A} \models G_\Sigma(N)$ implies $\mathcal{A} \models N$.*

Proof. Let $\mathcal{A}$ be a Herbrand model of $G_\Sigma(N)$. We have to show that $\mathcal{A} \models \forall\vec{x}\ C$ for all clauses $\forall\vec{x}\ C$ in N. This is equivalent to $\mathcal{A} \models C$, which in turn is equivalent to $\mathcal{A}(\beta)(C) = 1$ for all assignments β.

Choose $\beta : X \to U_\mathcal{A}$ arbitrarily. Since $\mathcal{A}$ is a Herbrand interpretation, $\beta(x)$ is a ground term for every variable x, so there is a substitution σ such that $x\sigma = \beta(x)$ for all variables x occurring in C. Now let γ be an arbitrary assignment, then for every variable occurring in C we have $(\gamma \circ \sigma)(x) = \mathcal{A}(\gamma)(x\sigma) = x\sigma = \beta(x)$ and consequently $\mathcal{A}(\beta)(C) = \mathcal{A}(\gamma \circ \sigma)(C) = \mathcal{A}(\gamma)(C\sigma)$. Since $C\sigma \in G_\Sigma(N)$ and $\mathcal{A}$ is a Herbrand model of $G_\Sigma(N)$, we get $\mathcal{A}(\gamma)(C\sigma) = 1$, so $\mathcal{A}$ is a model of C. □

Theorem 3.34 (Herbrand) *A set N of Σ-clauses is satisfiable if and only if it has a Herbrand model over Σ.*

Proof. The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let $N \not\models \bot$.

$$
\begin{aligned}
N \not\models \bot &\Rightarrow \bot \notin Res^*(N) && \text{(resolution is sound)}\\
&\Rightarrow \bot \notin G_\Sigma(Res^*(N))\\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models G_\Sigma(Res^*(N)) && \text{(Thm. 3.17; Cor. 3.30)}\\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models Res^*(N) && \text{(Lemma 3.33)}\\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models N && (N \subseteq Res^*(N)) \quad \square
\end{aligned}
$$

The Theorem of Löwenheim-Skolem

Theorem 3.35 (Löwenheim–Skolem) *Let Σ be a countable signature and let S be a set of closed Σ-formulas. Then S is satisfiable iff S has a model over a countable universe.*

Proof. If both X and Σ are countable, then S can be at most countably infinite. Now generate, maintaining satisfiability, a set N of clauses from S. This extends Σ by at most countably many new Skolem functions to Σ'. As Σ' is countable, so is $T_{\Sigma'}$, the universe of Herbrand-interpretations over Σ'. Now apply Theorem 3.34. □

Refutational Completeness of General Resolution

Theorem 3.36 *Let N be a set of general clauses where $Res(N) \subseteq N$. Then*

$$N \models \bot \text{ iff } \bot \in N.$$

Proof. Let $Res(N) \subseteq N$. By Corollary 3.30: $Res(G_\Sigma(N)) \subseteq G_\Sigma(N)$

$$\begin{aligned} N \models \bot &\Leftrightarrow G_\Sigma(N) \models \bot && \text{(Lemma 3.32/3.33; Theorem 3.34)} \\ &\Leftrightarrow \bot \in G_\Sigma(N) && \text{(propositional resolution sound and complete)} \\ &\Leftrightarrow \bot \in N \quad \square \end{aligned}$$

Compactness of Predicate Logic

Theorem 3.37 (Compactness Theorem for First-Order Logic) *Let S be a set of closed first-order formulas. S is unsatisfiable $\Leftrightarrow$ some finite subset $S' \subseteq S$ is unsatisfiable.*

Proof. The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let S be unsatisfiable and let N be the set of clauses obtained by Skolemization and CNF transformation of the formulas in S. Clearly $Res^*(N)$ is unsatisfiable. By Theorem 3.36, $\bot \in Res^*(N)$, and therefore $\bot \in Res^n(N)$ for some $n \in \mathbb{N}$. Consequently, $\bot$ has a finite resolution proof B of depth $\leq n$. Choose S' as the subset of formulas in S such that the corresponding clauses contain the assumptions (leaves) of B. $\square$

3.12 Ordered Resolution with Selection

Motivation: Search space for Res *very* large.

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem 3.17) one only needs to resolve and factor maximal atoms
   $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
   $\Rightarrow$ *ordering restrictions*

2. In the proof, it does not really matter with which negative literal an inference is performed
   $\Rightarrow$ choose a negative literal don't-care-nondeterministically
   $\Rightarrow$ *selection*

Ordering Restrictions

In the completeness proof one only needs to resolve and factor maximal atoms $\Rightarrow$ If we impose ordering restrictions on ground inferences, the proof remains correct:

Ordered Resolution:

$$\frac{D \vee A \qquad C \vee \neg A}{D \vee C}$$

if $A \succ L$ for all L in D and $\neg A \succeq L$ for all L in C.

Ordered Factorization:

$$\frac{C \vee A \vee A}{C \vee A}$$

if $A \succeq L$ for all L in C.

Problem: How to extend this to non-ground inferences?

In the completeness proof, we talk about (strictly) maximal literals of *ground* clauses.

In the non-ground calculus, we have to consider those literals that correspond to (strictly) maximal literals of ground instances.

An ordering $\succ$ on atoms (or terms) is called *stable under substitutions*, if $A \succ B$ implies $A\sigma \succ B\sigma$.

Note:

- We can not require that $A \succ B$ iff $A\sigma \succ B\sigma$.
- We can not require that $\succ$ is total on non-ground atoms.

Consequence: In the ordering restrictions for non-ground inferences, we have to replace $\succ$ by $\not\preceq$ and $\succeq$ by $\not\prec$.

Ordered Resolution:

$$\frac{D \vee B \qquad C \vee \neg A}{(D \vee C)\sigma}$$

if $\sigma = \mathrm{mgu}(A, B)$ and $B\sigma \not\preceq L\sigma$ for all L in D and $\neg A\sigma \not\prec L\sigma$ for all L in C.

Ordered Factorization:

$$\frac{C \vee A \vee B}{(C \vee A)\sigma}$$

if $\sigma = \mathrm{mgu}(A, B)$ and $A\sigma \not\prec L\sigma$ for all L in C.

Selection Functions

Selection functions can be used to override ordering restrictions for individual clauses.

A *selection function* is a mapping

$$\text{sel} : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \lor \neg A \lor B$$

$$\boxed{\neg B_0} \lor \boxed{\neg B_1} \lor A$$

Intuition:

- If a clause has at least one selected literal, compute only inferences that involve a selected literal.
- If a clause has no selected literals, compute only inferences that involve a maximal literal.

Resolution Calculus $Res^{\succ}_{\text{sel}}$

The resolution calculus $Res^{\succ}_{\text{sel}}$ is parameterized by

- a selection function sel
- and a well-founded ordering $\succ$ on atoms that is total on ground atoms and stable under substitutions.

$$\frac{D \vee B \qquad C \vee \neg A}{(D \vee C)\sigma} \qquad [\textit{ordered resolution with selection}]$$

if the following conditions are satisfied:

(i) $\sigma = \mathrm{mgu}(A, B)$;

(ii) $B\sigma$ strictly maximal in $D\sigma \vee B\sigma$, i. e., $B\sigma \not\preceq L\sigma$ for all L in D;

(iii) nothing is selected in $D \vee B$ by sel;

(iv) either $\neg A$ is selected, or nothing is selected in $C \vee \neg A$ and $\neg A\sigma$ is maximal in $C\sigma \vee \neg A\sigma$, i. e., $\neg A\sigma \not\prec L\sigma$ for all L in C.

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \qquad [\textit{ordered factorization}]$$

if the following conditions are satisfied:

(i) $\sigma = \mathrm{mgu}(A, B)$;

(ii) $A\sigma$ is maximal in $C\sigma \vee A\sigma \vee B\sigma$, i. e., $A\sigma \not\prec L\sigma$ for all L in C.

(iii) nothing is selected in $C \vee A \vee B$ by sel.

Special Case: $Res^{\succ}_{\mathrm{sel}}$ for Propositional Logic

For ground clauses the resolution inference rule simplifies to

$$\frac{D \vee A \qquad C \vee \neg A}{D \vee C}$$

if the following conditions are satisfied:

(i) $A \succ L$ for all L in D;

(ii) nothing is selected in $D \vee A$ by sel;

(iii) $\neg A$ is selected in $C \vee \neg A$, or nothing is selected in $C \vee \neg A$ and $\neg A \succeq L$ for all L in C.

Analogously, the factorization rule simplifies to

$$\frac{C \vee A \vee A}{C \vee A}$$

if the following conditions are satisfied:

(i) $A \succeq L$ for all L in C;

(ii) nothing is selected in $C \vee A \vee A$ by sel.

Search Spaces Become Smaller

1	$A \vee B$	
2	$A \vee \boxed{\neg B}$	
3	$\neg A \vee B$	
4	$\neg A \vee \boxed{\neg B}$	
5	$B \vee B$	Res 1, 3
6	B	Fact 5
7	$\neg A$	Res 6, 4
8	A	Res 6, 2
9	$\bot$	Res 8, 7

we assume $A \succ B$ and sel as indicated by $\boxed{X}$. The maximal literal in a clause is depicted in red.

With this ordering and selection function the refutation proceeds strictly deterministically in this example. Generally, proof search will still be non-deterministic but the search space will be much smaller than with unrestricted resolution.

Avoiding Rotation Redundancy

From

$$\dfrac{\dfrac{C_1 \vee A \quad C_2 \vee \neg A \vee B}{C_1 \vee C_2 \vee B} \qquad C_3 \vee \neg B}{C_1 \vee C_2 \vee C_3}$$

we can obtain by *rotation*

$$\dfrac{C_1 \vee A \qquad \dfrac{C_2 \vee \neg A \vee B \quad C_3 \vee \neg B}{C_2 \vee \neg A \vee C_3}}{C_1 \vee C_2 \vee C_3}$$

another proof of the same clause. In large proofs many rotations are possible. However, if $A \succ B$, then the second proof does not fulfill the ordering restrictions.

Conclusion: In the presence of ordering restrictions (however one chooses $\succ$) no rotations are possible. In other words, orderings identify exactly one representant in any class of rotation-equivalent proofs.

Lifting Lemma for $Res^{\succ}_{\text{sel}}$

Lemma 3.38 *Let D and C be variable-disjoint clauses. If*

$$\begin{array}{ccc} D & & C \\ \downarrow \sigma & & \downarrow \rho \\ D\sigma & & C\rho \\ \hline & C' & \end{array} \qquad \textit{[propositional inference in } Res^{\succ}_{\text{sel}}\textit{]}$$

and if $\text{sel}(D\sigma) \simeq \text{sel}(D)$, $\text{sel}(C\rho) \simeq \text{sel}(C)$ *(that is, "corresponding" literals are selected), then there exists a substitution τ such that*

$$\begin{array}{c} \dfrac{D \qquad C}{C''} \\ \downarrow \tau \\ C' = C''\tau \end{array} \qquad \textit{[inference in } Res^{\succ}_{\text{sel}}\textit{]}$$

An analogous lifting lemma holds for factorization.

Saturation of Sets of General Clauses

Corollary 3.39 *Let N be a set of general clauses saturated under $Res^{\succ}_{\text{sel}}$, i. e., $Res^{\succ}_{\text{sel}}(N) \subseteq N$. Then there exists a selection function* sel' *such that* $\text{sel}|_N = \text{sel}'|_N$ *and $G_\Sigma(N)$ is also saturated, i. e.,*

$$Res^{\succ}_{\text{sel}'}(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Proof. We first define the selection function sel' such that $\text{sel}'(C) = \text{sel}(C)$ for all clauses $C \in G_\Sigma(N) \cap N$. For $C \in G_\Sigma(N) \setminus N$ we choose a fixed but arbitrary clause $D \in N$ with $C \in G_\Sigma(D)$ and define $\text{sel}'(C)$ to be those occurrences of literals that are ground instances of the occurrences selected by sel in D. Then proceed as in the proof of Cor. 3.30 using the above lifting lemma. □

Soundness and Refutational Completeness

Theorem 3.40 *Let $\succ$ be an atom ordering and* sel *a selection function such that $Res^{\succ}_{\text{sel}}(N) \subseteq N$. Then*

$$N \models \bot \Leftrightarrow \bot \in N$$

Proof. The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part consider first the propositional level: Construct a candidate interpretation I_N as for unrestricted resolution, except that clauses C in N that have selected literals are not productive, even if they are false in I_C and if their maximal atom occurs only once and is positive. The result for general clauses follows using Corollary 3.39. □