2.5 Improving the CNF Transformation

The goal

"find a formula G in CNF such that $F \models\mid G$"

is unpractical.

But if we relax the requirement to

"find a formula G in CNF such that $F \models \bot \Leftrightarrow G \models \bot$"

we can get an efficient transformation.

Tseitin Transformation

Proposition 2.12 *A formula $H[F]_p$ is satisfiable if and only if $H[Q]_p \wedge (Q \leftrightarrow F)$ is satisfiable, where Q is a new propositional variable that works as an abbreviation for F.*

Satisfiability-preserving CNF transformation (Tseitin 1970):

Use the rule above recursively for all subformulas in the original formula (this introduces a linear number of new propositional variables Q and definitions $Q \leftrightarrow F$).

Convert of the resulting conjunction to CNF (this increases the size only by an additional factor, since each formula $Q \leftrightarrow F$ yields at most four clauses in the CNF).

Polarity-based CNF Transformation

A further improvement is possible by taking the *polarity* of the subformula F into account.

Proposition 2.13 *Let $\mathcal{A}$ be a valuation, let F and G be formulas, and let $H = H[F]_p$ be a formula in which F occurs as a subformula at position p.*

If $\mathrm{pol}(H,p) = 1$ *and* $\mathcal{A}(F) \leq \mathcal{A}(G)$, *then* $\mathcal{A}(H[F]_p) \leq \mathcal{A}(H[G]_p)$.

If $\mathrm{pol}(H,p) = -1$ *and* $\mathcal{A}(F) \geq \mathcal{A}(G)$, *then* $\mathcal{A}(H[F]_p) \leq \mathcal{A}(H[G]_p)$.

Proof. Exercise. □

Let Q be a propositional variable not occurring in $H[F]_p$.

Define the formula $\mathrm{def}(H, p, Q, F)$ by

- $(Q \to F)$, if $\mathrm{pol}(H, p) = 1$,
- $(F \to Q)$, if $\mathrm{pol}(H, p) = -1$,
- $(Q \leftrightarrow F)$, if $\mathrm{pol}(H, p) = 0$.

Proposition 2.14 *Let Q be a propositional variable not occurring in $H[F]_p$. Then $H[F]_p$ is satisfiable if and only if $H[Q]_p \land \mathrm{def}(H, p, Q, F)$ is satisfiable.*

Proof. ($\Rightarrow$) Since $H[F]_p$ is satisfiable, there exists a Π-valuation $\mathcal{A}$ such that $\mathcal{A} \models H[F]_p$. Let $\Pi' = \Pi \cup \{Q\}$ and define the Π'-valuation $\mathcal{A}'$ by $\mathcal{A}'(P) = \mathcal{A}(P)$ for $P \in \Pi$ and $\mathcal{A}'(Q) = \mathcal{A}(F)$. Obviously $\mathcal{A}'(\mathrm{def}(H, p, Q, F)) = 1$; moreover $\mathcal{A}'(H[Q]_p) = \mathcal{A}'(H[F]_p) = \mathcal{A}(H[F]_p) = 1$ by Prop. 2.8, so $H[Q]_p \land \mathrm{def}(H, p, Q, F)$ is satisfiable.

($\Leftarrow$) Let $\mathcal{A}$ be a valuation such that $\mathcal{A} \models H[Q]_p \land \mathrm{def}(H, p, Q, F)$. So $\mathcal{A}(H[Q]_p) = 1$ and $\mathcal{A}(\mathrm{def}(H, p, Q, F)) = 1$. We will show that $\mathcal{A} \models H[F]_p$.

If $\mathrm{pol}(H, p) = 0$, then $\mathrm{def}(H, p, Q, F) = (Q \leftrightarrow F)$, so $\mathcal{A}(Q) = \mathcal{A}(F)$, hence $\mathcal{A}(H[F]_p) = \mathcal{A}(H[Q]_p) = 1$ by Prop. 2.8.

If $\mathrm{pol}(H, p) = 1$, then $\mathrm{def}(H, p, Q, F) = (Q \to F)$, so $\mathcal{A}(Q) \leq \mathcal{A}(F)$. By Prop. 2.13, $\mathcal{A}(H[F]_p) \geq \mathcal{A}(H[Q]_p) = 1$, so $\mathcal{A}(H[F]_p) = 1$.

If $\mathrm{pol}(H, p) = -1$, then $\mathrm{def}(H, p, Q, F) = (F \to Q)$, so $\mathcal{A}(F) \leq \mathcal{A}(Q)$. By Prop. 2.13, $\mathcal{A}(H[F]_p) \geq \mathcal{A}(H[Q]_p) = 1$, so $\mathcal{A}(H[F]_p) = 1$. $\square$

Optimized CNF

Not every introduction of a definition for a subformula leads to a smaller CNF.

The number of eventually generated clauses is a good indicator for useful CNF transformations.

The functions ν and $\bar{\nu}$ give us an overapproximation for the number of clauses generated by a formula that occurs positively/negatively.

G	$\nu(G)$	$\bar{\nu}(G)$
$P, \top, \bot$	1	1
$F_1 \wedge F_2$	$\nu(F_1) + \nu(F_2)$	$\bar{\nu}(F_1)\bar{\nu}(F_2)$
$F_1 \vee F_2$	$\nu(F_1)\nu(F_2)$	$\bar{\nu}(F_1) + \bar{\nu}(F_2)$
$\neg F_1$	$\bar{\nu}(F_1)$	$\nu(F_1)$
$F_1 \rightarrow F_2$	$\bar{\nu}(F_1)\nu(F_2)$	$\nu(F_1) + \bar{\nu}(F_2)$
$F_1 \leftrightarrow F_2$	$\nu(F_1)\bar{\nu}(F_2) + \bar{\nu}(F_1)\nu(F_2)$	$\nu(F_1)\nu(F_2)+\bar{\nu}(F_1)\bar{\nu}(F_2)$

A better CNF transformation:

Step 1: Exhaustively apply modulo commutativity of $\leftrightarrow$ and associativity/commutativity of $\wedge$, $\vee$:

$$
\begin{aligned}
H[(F \wedge \top)]_p &\Rightarrow_{\mathrm{OCNF}} H[F]_p \\
H[(F \vee \bot)]_p &\Rightarrow_{\mathrm{OCNF}} H[F]_p \\
H[(F \leftrightarrow \bot)]_p &\Rightarrow_{\mathrm{OCNF}} H[\neg F]_p \\
H[(F \leftrightarrow \top)]_p &\Rightarrow_{\mathrm{OCNF}} H[F]_p \\
H[(F \vee \top)]_p &\Rightarrow_{\mathrm{OCNF}} H[\top]_p \\
H[(F \wedge \bot)]_p &\Rightarrow_{\mathrm{OCNF}} H[\bot]_p
\end{aligned}
$$

$$
\begin{aligned}
H[(F \wedge F)]_p &\Rightarrow_{\mathrm{OCNF}} H[F]_p \\
H[(F \vee F)]_p &\Rightarrow_{\mathrm{OCNF}} H[F]_p \\
H[(F \wedge (F \vee G))]_p &\Rightarrow_{\mathrm{OCNF}} H[F]_p \\
H[(F \vee (F \wedge G))]_p &\Rightarrow_{\mathrm{OCNF}} H[F]_p \\
H[(F \wedge \neg F)]_p &\Rightarrow_{\mathrm{OCNF}} H[\bot]_p \\
H[(F \vee \neg F)]_p &\Rightarrow_{\mathrm{OCNF}} H[\top]_p \\
H[\neg \top]_p &\Rightarrow_{\mathrm{OCNF}} H[\bot]_p \\
H[\neg \bot]_p &\Rightarrow_{\mathrm{OCNF}} H[\top]_p
\end{aligned}
$$

$$
\begin{aligned}
H[(F \rightarrow \bot)]_p &\Rightarrow_{\mathrm{OCNF}} H[\neg F]_p \\
H[(F \rightarrow \top)]_p &\Rightarrow_{\mathrm{OCNF}} H[\top]_p \\
H[(\bot \rightarrow F)]_p &\Rightarrow_{\mathrm{OCNF}} H[\top]_p \\
H[(\top \rightarrow F)]_p &\Rightarrow_{\mathrm{OCNF}} H[F]_p
\end{aligned}
$$

Step 2: Introduce top-down fresh variables for beneficial subformulas:

$$H[F]_p \;\Rightarrow_{\mathrm{OCNF}}\; H[P]_p \wedge \operatorname{def}(H, p, P, F)$$

where P is new to $H[F]_p$ and $\nu(H[F]_p) > \nu(H[P]_p \wedge \operatorname{def}(H, p, P, F))$.

Remark: Although computing ν is not practical in general, the test $\nu(H[F]_p) > \nu(H[P]_p \wedge \operatorname{def}(H, p, P, F))$ can be computed in constant time.

Step 3: Eliminate equivalences dependent on their polarity:

$$H[F \leftrightarrow G]_p \;\Rightarrow_{\mathrm{OCNF}}\; H[(F \rightarrow G) \wedge (G \rightarrow F)]_p$$

if $\operatorname{pol}(F, p) = 1$ or $\operatorname{pol}(F, p) = 0$.

$$H[F \leftrightarrow G]_p \;\Rightarrow_{\mathrm{OCNF}}\; H[(F \wedge G) \vee (\neg F \wedge \neg G)]_p$$

if $\operatorname{pol}(F, p) = -1$.

Step 4: Apply steps 2, 3, 4, 5 of $\Rightarrow_{\mathrm{CNF}}$

Remark: The $\Rightarrow_{\mathrm{OCNF}}$ algorithm is already close to a state of the art algorithm, but some additional redundancy tests and simplification mechanisms are missing.