Assignment 1 *(Abstract Reduction Systems)* (4 + 6 + 6 = 16 points)

Let $(A, \to)$ be an abstract reduction system such that every element of A has exactly one normal form w.r.t. $\to$. For every $b \in A$ define $L(b)$ as the minimal $n \in \mathbb{N}$ such that $b \to^n b'$ and b' is in normal form w.r.t. $\to$. Define the binary relation $\Rightarrow$ over A by $b \Rightarrow c$ if and only if $b \to c$ and $L(b) > L(c)$.

Part (a) Give an example that shows that $\to \neq \Rightarrow$.

Part (b) Show that for every $b \in A$ we have $b \Rightarrow^* b'$, where b' is the normal form of b w.r.t. $\to$.

Part (c) Use part (b) to show that $\leftrightarrow^* = \Leftrightarrow^*$.

Assignment 2 *(Algebras, Herbrand Models)* (10 points)

Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{b/0,\ f/1\}$ and $\Pi = \{P/1\}$. Let F be the Σ-formula

$$\neg P(b) \wedge P(f(f(b))) \wedge \forall x\, (\neg P(x) \vee P(f(x))).$$

Decide for each of the following statements whether they are true or false:

(1) There is a Σ-model $\mathcal{A}$ of F such that $U_{\mathcal{A}} = \{7, 8, 9\}$.

(2) There is a Σ-model $\mathcal{A}$ of F such that $f_{\mathcal{A}}(a) = a$ for every $a \in U_{\mathcal{A}}$.

(3) F has exactly four Σ-models.

(4) Every Σ-model of F is a model of $\exists x\, P(x)$.

(5) Every Σ-model of F is a model of $\forall x\, P(f(f(x)))$.

(6) There are infinitely many Herbrand interpretations over Σ.

(7) There is a Herbrand model of F over Σ whose universe has exactly two elements.

(8) There is a Herbrand model of F over Σ with an infinite universe.

(9) F has exactly two Herbrand models over Σ.

(Note on grading: A yes/no answer is sufficient; you do not have to give an explanation. However, you need at least five correct answers to get any points for this assignment. Missing answers count like false answers.)

Assignment 3 *(Algebras)* (14 points)

Let $\Sigma = (\Omega, \emptyset)$, let $\mathcal{A}$ and $\mathcal{B}$ be Σ-algebras. A function $\phi : U_{\mathcal{A}} \to U_{\mathcal{B}}$ is called a Σ-homomorphism, if for all $f/n \in \Omega$ and all $a_1, \dots, a_n \in U_{\mathcal{A}}$, we have $\phi(f_{\mathcal{A}}(a_1, \dots, a_n)) = f_{\mathcal{B}}(\phi(a_1), \dots, \phi(a_n))$. Let $\forall \vec{x}(s \approx t)$ be a universally quantified Σ-equation. Prove: If $\phi : U_{\mathcal{A}} \to U_{\mathcal{B}}$ is a surjective (i.e., onto) Σ-homomorphism, and $\mathcal{A} \models \forall \vec{x}(s \approx t)$, then $\mathcal{B} \models \forall \vec{x}(s \approx t)$.

(Note: Even though this was not explicitly said during the lecture, you may use Lemma 3.21 also for first-order logic *with* equality.)

Assignment 4 *(Redundancy)* (12 points)

Let $\Sigma = (\{f/1,\ b/0,\ c/0\},\ \{P/1\})$; let N be the following set of Σ-clauses:

$$P(b) \qquad (1)$$
$$P(f(c)) \qquad (2)$$
$$\neg P(x) \vee P(f(x)) \qquad (3)$$

Let $\succ$ be a Knuth-Bendix ordering with weight 1 for all function and predicate symbols and variables and the precedence $P > f > b > c$. The ordering is extended to ground literals and ground clauses as usual. Give the smallest non-empty ground Σ-clauses C_1, C_2, C_3, C_4 such that

(a) $C_1 \in G_\Sigma(N)$ and $C_1 \in \mathrm{Red}(N)$,

(b) $C_2 \in G_\Sigma(N)$ and $C_2 \notin \mathrm{Red}(N)$,

(c) $C_3 \notin G_\Sigma(N)$ and $C_3 \in \mathrm{Red}(N)$,

(d) $C_4 \notin G_\Sigma(N)$ and $C_4 \notin \mathrm{Red}(N)$.

Assignment 5 *(Tableaux)* (7 + 7 = 14 points)

Check whether the following propositional formulas are valid or not using semantic tableaux. Give a brief explanation. (Use exactly the expansion rules given in the lecture. Do not use shortcuts; do not convert the formulas to NNF, CNF, or DNF.)

Part (a)

$$(P \rightarrow Q) \rightarrow ((P \vee R) \rightarrow (Q \vee R))$$

Part (b)

$$(P \vee Q) \rightarrow (P \wedge Q)$$

Assignment 6 *(Knuth-Bendix Completion)* (14 points)

Let E be the following set of equations over $\Sigma = (\{f/2,\ g/1,\ h/1,\ b/0\}, \emptyset)$.

$$f(g(x), x) \approx b \qquad (1)$$
$$f(x, b) \approx x \qquad (2)$$
$$g(h(x)) \approx x \qquad (3)$$

Apply the Knuth-Bendix completion procedure to E and transform it into a finite convergent term rewrite system; use a Knuth-Bendix ordering with weight 1 for all function symbols and variables and the precedence $f > g > h > b$. Use a reasonable strategy.