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Tutorials for “Automated Reasoning”
Exercise sheet 11

Exercise 11.1: *(5 P)*
Let $E = \{\, f(g(x)) \approx g(f(x)) \,\}$. Give a derivation for $E \vdash f(f(g(g(y)))) \approx g(g(f(f(y))))$.

Exercise 11.2: *(5 P)*
Let $\Sigma = (\{f/2, b/0, c/0, d/0\}, \emptyset)$; let $E = \{\forall x\, (f(x,x) \approx b),\ c \approx d\}$; let $X = \{x, y, z\}$ be a set of variables. For any $t \in \mathrm{T}_\Sigma(X)$ let $[t]$ denote the congruence class of t w.r.t. E. Let $\mathcal{T} = \mathrm{T}_\Sigma(X)/E$ and let $\beta : X \to \mathcal{T}$ be the assignment that maps every variable to $[c]$.

State for each of the following “items” whether it is (a) a term, (b) a set of terms, (c) a set of ground terms, (d) a congruence class w.r.t. E, (e) a formula, (f) a truth value, or (g) non-sensical. (More than one answer may be correct.)

(1)	$[c]$	(6)	$f([c],[d])$
(2)	$[b]$	(7)	$f_{\mathcal{T}}(\{c,d\},\{c,d\})$
(3)	$\{d\}$	(8)	$b \approx c$
(4)	$f(c,d)$	(9)	$\mathcal{T}(\beta)(\forall x, y\ (f(x,x) \approx f(y,y)))$
(5)	$[f(c,d)]$	(10)	$f_{\mathcal{T}}(\mathcal{T}(\beta)(x), \mathcal{T}(\beta)(y))$

Exercise 11.3: *(4 P)*
Is the rewrite system

$$\{\, f(a) \to f(b),\ f(b) \to f(c),\ f(c) \to f(a),\ f(x) \to x \,\}$$

(i) terminating, (ii) normalizing, (iii) locally confluent, (iv) confluent? Give a brief explanation.

Exercise 11.4: (*5 P*)
A friend asks you to proofread his master thesis. On page 15 of the thesis, your friend writes the following:

> **Lemma 5.** Let $\succ$ be a noetherian ordering over a set A, let $\rightarrow$ be a binary relation such that $\rightarrow\ \subseteq\ \succ$. Let s and r be two elements of A, such that r is irreducible with respect to $\rightarrow$, and define $A_r^s = \{\, t \in A \mid s \succeq t,\ t \rightarrow^* r \,\}$. If for every $t_0, t_1, t_2 \in A$ such that $s \succeq t_0$ and $t_1 \leftarrow t_0 \rightarrow t_2 \rightarrow^* r$ there exists a $t_3 \in A$ such that $t_1 \rightarrow^* t_3 \leftarrow^* t_2$, then for every $t_0 \in A_r^s$ and $t_1' \in A$, $t_0 \rightarrow^* t_1'$ implies $t_1' \in A_r^s$.
>
> **Proof.** We use well-founded induction over t_0 with respect to $\succ$. Let $t_0 \in A_r^s$ and $t_1' \in A$ such that $t_0 \rightarrow^* t_1'$. If this derivation is empty, the result is trivial, so suppose that $t_0 \rightarrow t_1 \rightarrow^* t_1'$. As $t_0 \in A_r^s$ is reducible, it is different from r, hence there is a non-empty derivation $t_0 \rightarrow t_2 \rightarrow^* r$. By assumption, there exists a $t_3 \in A$ such that $t_1 \rightarrow^* t_3 \leftarrow^* t_2$. Now $t_0 \succ t_2$ and $t_2 \in A_r^s$, hence $t_3 \in A_r^s$ by the induction hypothesis, and thus $t_1 \in A_r^s$. Since $t_0 \succ t_1$, we can use the induction hypothesis once more and obtain $t_1' \in A_r^s$ as required.

(1) Is the "proof" correct (yes/no)?

(2) If the "proof" is not correct:

   (a) Which step is incorrect?

   (b) Does the "theorem" hold? If yes, give a correct proof, otherwise give a counterexample.

Challenge Problem: (*6 Bonus Points*)
Find a signature Σ containing at least one constant symbol, a set E of Σ-equations, and two terms $s, t \in \mathrm{T}_\Sigma(X)$ such that

$$\mathrm{T}_\Sigma(\{x_1\})/E \models \forall\vec{x}(s \approx t),$$

but

$$\mathrm{T}_\Sigma(\{x_1, x_2\})/E \not\models \forall\vec{x}(s \approx t)$$

where $\vec{x}$ consists of all the variables occurring in s and t. (The variables in $\vec{x}$ need not be contained in $\{x_1, x_2\}$.)

Submit your solution in lecture hall E1.3, Room 003 during the lecture on January 25. Please write your name and the time of your tutorial group (Mo 8–10 or Mo 12–14) on your solution.

Joint solutions, prepared by up to three persons together, are allowed (but not encouraged). If you prepare your solution jointly, submit it only once and indicate all authors on the sheet.