4.3 Confluence

Let $(A, \to)$ be an abstract reduction system.

b and $c \in A$ are *joinable*, if there is a a such that $b \to^* a \leftarrow^* c$.
Notation: $b \downarrow c$.

The relation $\to$ is called

Church-Rosser, if $b \leftrightarrow^* c$ implies $b \downarrow c$.

confluent, if $b \leftarrow^* a \to^* c$ implies $b \downarrow c$.

locally confluent, if $b \leftarrow a \to c$ implies $b \downarrow c$.

convergent, if it is confluent and terminating.

Theorem 4.7 *The following properties are equivalent:*

(i) $\to$ *has the Church-Rosser property.*

(ii) $\to$ *is confluent.*

Proof. (i)⇒(ii): trivial.

(ii)⇒(i): by induction on the number of peaks in the derivation $b \leftrightarrow^* c$. □

Lemma 4.8 *If* $\to$ *is confluent, then every element has at most one normal form.*

Proof. Suppose that some element $a \in A$ has normal forms b and c, then $b \leftarrow^* a \to^* c$. If $\to$ is confluent, then $b \to^* d \leftarrow^* c$ for some $d \in A$. Since b and c are normal forms, both derivations must be empty, hence $b \to^0 d \leftarrow^0 c$, so b, c, and d must be identical. □

Corollary 4.9 *If* $\to$ *is normalizing and confluent, then every element* b *has a unique normal form.*

Proposition 4.10 *If* $\to$ *is normalizing and confluent, then* $b \leftrightarrow^* c$ *if and only if* $b\downarrow = c\downarrow$.

Proof. Either using Thm. 4.7 or directly by induction on the length of the derivation of $b \leftrightarrow^* c$. □

Confluence and Local Confluence

Theorem 4.11 ("Newman's Lemma") *If a terminating relation $\rightarrow$ is locally confluent, then it is confluent.*

Proof. Let $\rightarrow$ be a terminating and locally confluent relation. Then $\rightarrow^+$ is a well-founded ordering. Define $\phi(a) \ \Leftrightarrow \ \big(\forall b, c : b \leftarrow^* a \rightarrow^* c \Rightarrow b \downarrow c\big)$.

We prove $\phi(a)$ for all $a \in A$ by well-founded induction over $\rightarrow^+$:

Case 1: $b \leftarrow^0 a \rightarrow^* c$: trivial.

Case 2: $b \leftarrow^* a \rightarrow^0 c$: trivial.

Case 3: $b \leftarrow^* b' \leftarrow a \rightarrow c' \rightarrow^* c$: use local confluence, then use the induction hypothesis. □

Rewrite Relations

Corollary 4.12 *If E is convergent (i. e., terminating and confluent), then $s \approx_E t$ if and only if $s \leftrightarrow_E^* t$ if and only if $s{\downarrow_E} = t{\downarrow_E}$.*

Corollary 4.13 *If E is finite and convergent, then $\approx_E$ is decidable.*

Reminder:
If E is terminating, then it is confluent if and only if it is locally confluent.

Problems:

Show local confluence of E.

Show termination of E.

Transform E into an equivalent set of equations that is locally confluent and terminating.

4.4 Critical Pairs

Showing local confluence (Sketch):

Problem: If $t_1 \leftarrow_E t_0 \rightarrow_E t_2$, does there exist a term s such that $t_1 \rightarrow_E^* s \leftarrow_E^* t_2$?

If the two rewrite steps happen in different subtrees (disjoint redexes): yes.

If the two rewrite steps happen below each other (overlap at or below a variable position): yes.

If the left-hand sides of the two rules overlap at a non-variable position: needs further investigation.

Question:
Are there rewrite rules $l_1 \rightarrow r_1$ and $l_2 \rightarrow r_2$ such that some subterm $l_1|_p$ and l_2 have a common instance $(l_1|_p)\sigma_1 = l_2\sigma_2$?

Observation:
If we assume w.o.l.o.g. that the two rewrite rules do not have common variables, then only a single substitution is necessary: $(l_1|_p)\sigma = l_2\sigma$.

Further observation:
The mgu of $l_1|_p$ and l_2 subsumes all unifiers σ of $l_1|_p$ and l_2.

Let $l_i \rightarrow r_i$ $(i = 1, 2)$ be two rewrite rules in a TRS R whose variables have been renamed such that $\text{var}(l_1) \cap \text{var}(l_2) = \emptyset$. (Remember that $\text{var}(l_i) \supseteq \text{var}(r_i)$.)

Let $p \in \text{pos}(l_1)$ be a position such that $l_1|_p$ is not a variable and σ is an mgu of $l_1|_p$ and l_2.

Then $r_1\sigma \leftarrow l_1\sigma \rightarrow (l_1\sigma)[r_2\sigma]_p$.

$\langle r_1\sigma, (l_1\sigma)[r_2\sigma]_p \rangle$ is called a *critical pair* of R.

The critical pair is *joinable* (or: converges), if $r_1\sigma \downarrow_R (l_1\sigma)[r_2\sigma]_p$.

Theorem 4.14 ("Critical Pair Theorem") *A TRS R is locally confluent if and only if all its critical pairs are joinable.*

Proof. "only if": obvious, since joinability of a critical pair is a special case of local confluence.

"if": Suppose s rewrites to t_1 and t_2 using rewrite rules $l_i \rightarrow r_i \in R$ at positions $p_i \in \text{pos}(s)$, where $i = 1, 2$. Without loss of generality, we can assume that the two rules are variable disjoint, hence $s|_{p_i} = l_i\theta$ and $t_i = s[r_i\theta]_{p_i}$.

We distinguish between two cases: Either p_1 and p_2 are in disjoint subtrees ($p_1 \parallel p_2$), or one is a prefix of the other (w.o.l.o.g., $p_1 \leq p_2$).

Case 1: $p_1 \parallel p_2$.

Then $s = s[l_1\theta]_{p_1}[l_2\theta]_{p_2}$, and therefore $t_1 = s[r_1\theta]_{p_1}[l_2\theta]_{p_2}$ and $t_2 = s[l_1\theta]_{p_1}[r_2\theta]_{p_2}$.

Let $t_0 = s[r_1\theta]_{p_1}[r_2\theta]_{p_2}$. Then clearly $t_1 \to_R t_0$ using $l_2 \to r_2$ and $t_2 \to_R t_0$ using $l_1 \to r_1$.

Case 2: $p_1 \leq p_2$.

Case 2.1: $p_2 = p_1q_1q_2$, where $l_1|_{q_1}$ is some variable x.

In other words, the second rewrite step takes place at or below a variable in the first rule. Suppose that x occurs m times in l_1 and n times in r_1 (where $m \geq 1$ and $n \geq 0$).

Then $t_1 \to_R^* t_0$ by applying $l_2 \to r_2$ at all positions $p_1q'q_2$, where q' is a position of x in r_1.

Conversely, $t_2 \to_R^* t_0$ by applying $l_2 \to r_2$ at all positions p_1qq_2, where q is a position of x in l_1 different from q_1, and by applying $l_1 \to r_1$ at p_1 with the substitution θ', where $\theta' = \theta[x \mapsto (x\theta)[r_2\theta]_{q_2}]$.

Case 2.2: $p_2 = p_1p$, where p is a non-variable position of l_1.

Then $s|_{p_2} = l_2\theta$ and $s|_{p_2} = (s|_{p_1})|_p = (l_1\theta)|_p = (l_1|_p)\theta$, so θ is a unifier of l_2 and $l_1|_p$.

Let σ be the mgu of l_2 and $l_1|_p$, then $\theta = \tau \circ \sigma$ and $\langle r_1\sigma, (l_1\sigma)[r_2\sigma]_p\rangle$ is a critical pair.

By assumption, it is joinable, so $r_1\sigma \to_R^* v \leftarrow_R^* (l_1\sigma)[r_2\sigma]_p$.

Consequently, $t_1 = s[r_1\theta]_{p_1} = s[r_1\sigma\tau]_{p_1} \to_R^* s[v\tau]_{p_1}$ and $t_2 = s[r_2\theta]_{p_2} = s[(l_1\theta)[r_2\theta]_p]_{p_1} = s[(l_1\sigma\tau)[r_2\sigma\tau]_p]_{p_1} = s[((l_1\sigma)[r_2\sigma]_p)\tau]_{p_1} \to_R^* s[v\tau]_{p_1}$.

This completes the proof of the Critical Pair Theorem. □

Note: Critical pairs between a rule and (a renamed variant of) itself must be considered – except if the overlap is at the root (i. e., $p = \varepsilon$).

Corollary 4.15 *A terminating TRS R is confluent if and only if all its critical pairs are joinable.*

Proof. By Newman's Lemma and the Critical Pair Theorem. □

Corollary 4.16 *For a finite terminating TRS, confluence is decidable.*

Proof. For every pair of rules and every non-variable position in the first rule there is at most one critical pair $\langle u_1, u_2\rangle$.

Reduce every u_i to some normal form u_i'. If $u_1' = u_2'$ for every critical pair, then R is confluent, otherwise there is some non-confluent situation $u_1' \leftarrow_R^* u_1 \leftarrow_R s \to_R u_2 \to_R^* u_2'$. □

4.5 Termination

Termination problems:

Given a finite TRS R and a term t, are all R-reductions starting from t terminating?

Given a finite TRS R, are all R-reductions terminating?

Proposition 4.17 *Both termination problems for TRSs are undecidable in general.*

Proof. Encode Turing machines using rewrite rules and reduce the (uniform) halting problems for TMs to the termination problems for TRSs. □

Consequence:

Decidable criteria for termination are not complete.

Two Different Scenarios

Depending on the application, the TRS whose termination we want to show can be

(i) fixed and known in advance, or

(ii) evolving (e.g., generated by some saturation process).

Methods for case (ii) are also usable for case (i).
Many methods for case (i) are not usable for case (ii).

We will first consider case (ii);
additional techniques for case (i) will be considered later.

Reduction Orderings

Goal:

Given a finite TRS R, show termination of R by looking at finitely many rules $l \rightarrow r \in R$, rather than at infinitely many possible replacement steps $s \rightarrow_R s'$.

A binary relation $\sqsupset$ over $\mathrm{T}_\Sigma(X)$ is called *compatible with Σ-operations,* if $s \sqsupset s'$ implies $f(t_1, \dots, s, \dots, t_n) \sqsupset f(t_1, \dots, s', \dots, t_n)$ for all $f \in \Omega$ and $s, s', t_i \in \mathrm{T}_\Sigma(X)$.

Lemma 4.18 *The relation $\sqsupset$ is compatible with Σ-operations, if and only if $s \sqsupset s'$ implies $t[s]_p \sqsupset t[s']_p$ for all $s, s', t \in \mathrm{T}_\Sigma(X)$ and $p \in \mathrm{pos}(t)$.*

Note: *compatible with Σ-operations = compatible with contexts.*

A binary relation $\sqsupset$ over $\mathrm{T}_\Sigma(X)$ is called *stable under substitutions,* if $s \sqsupset s'$ implies $s\sigma \sqsupset s'\sigma$ for all $s, s' \in \mathrm{T}_\Sigma(X)$ and substitutions σ.

A binary relation $\sqsupset$ is called a *rewrite relation,* if it is compatible with Σ-operations and stable under substitutions.

Example: If R is a TRS, then $\rightarrow_R$ is a rewrite relation.

A strict partial ordering over $\mathrm{T}_\Sigma(X)$ that is a rewrite relation is called *rewrite ordering.*

A well-founded rewrite ordering is called *reduction ordering.*

Theorem 4.19 *A TRS R terminates if and only if there exists a reduction ordering $\succ$ such that $l \succ r$ for every rule $l \rightarrow r \in R$.*

Proof. "if": $s \rightarrow_R s'$ if and only if $s = t[l\sigma]_p$, $s' = t[r\sigma]_p$. If $l \succ r$, then $l\sigma \succ r\sigma$ and therefore $t[l\sigma]_p \succ t[r\sigma]_p$. This implies $\rightarrow_R \subseteq \succ$. Since $\succ$ is a well-founded ordering, $\rightarrow_R$ is terminating.

"only if": Define $\succ = \rightarrow_R^+$. If $\rightarrow_R$ is terminating, then $\succ$ is a reduction ordering. □

The Interpretation Method

Proving termination by interpretation:

Let $\mathcal{A}$ be a Σ-algebra; let $\succ$ be a well-founded strict partial ordering on its universe.

Define the ordering $\succ_\mathcal{A}$ over $\mathrm{T}_\Sigma(X)$ by $s \succ_\mathcal{A} t$ iff $\mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(t)$ for all assignments $\beta : X \rightarrow U_\mathcal{A}$.

Is $\succ_\mathcal{A}$ a reduction ordering?

Lemma 4.20 *$\succ_\mathcal{A}$ is stable under substitutions.*

Proof. Let $s \succ_\mathcal{A} s'$, that is, $\mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(s')$ for all assignments $\beta : X \rightarrow U_\mathcal{A}$. Let σ be a substitution. We have to show that $\mathcal{A}(\gamma)(s\sigma) \succ \mathcal{A}(\gamma)(s'\sigma)$ for all assignments $\gamma : X \rightarrow U_\mathcal{A}$. Choose $\beta = \gamma \circ \sigma$, then by the substitution lemma, $\mathcal{A}(\gamma)(s\sigma) = \mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(s') = \mathcal{A}(\gamma)(s'\sigma)$. Therefore $s\sigma \succ_\mathcal{A} s'\sigma$. □

A function $f : U_\mathcal{A}^n \rightarrow U_\mathcal{A}$ is called *monotone* (with respect to $\succ$), if $a \succ a'$ implies $f(b_1, \dots, a, \dots, b_n) \succ f(b_1, \dots, a', \dots, b_n)$ for all $a, a', b_i \in U_\mathcal{A}$.

Lemma 4.21 *If the interpretation $f_\mathcal{A}$ of every function symbol f is monotone w. r. t. $\succ$, then $\succ_\mathcal{A}$ is compatible with Σ-operations.*

Proof. Let $s \succ s'$, that is, $\mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(s')$ for all $\beta : X \to U_{\mathcal{A}}$. Let $\beta : X \to U_{\mathcal{A}}$ be an arbitrary assignment. Then

$$\begin{aligned}\mathcal{A}(\beta)(f(t_1, \dots, s, \dots, t_n)) &= f_{\mathcal{A}}(\mathcal{A}(\beta)(t_1), \dots, \mathcal{A}(\beta)(s), \dots, \mathcal{A}(\beta)(t_n)) \\ &\succ f_{\mathcal{A}}(\mathcal{A}(\beta)(t_1), \dots, \mathcal{A}(\beta)(s'), \dots, \mathcal{A}(\beta)(t_n)) \\ &= \mathcal{A}(\beta)(f(t_1, \dots, s', \dots, t_n))\end{aligned}$$

Therefore $f(t_1, \dots, s, \dots, t_n) \succ_{\mathcal{A}} f(t_1, \dots, s', \dots, t_n)$. □

Theorem 4.22 *If the interpretation $f_{\mathcal{A}}$ of every function symbol f is monotone w. r. t. $\succ$, then $\succ_{\mathcal{A}}$ is a reduction ordering.*

Proof. By the previous two lemmas, $\succ_{\mathcal{A}}$ is a rewrite relation. If there were an infinite chain $s_1 \succ_{\mathcal{A}} s_2 \succ_{\mathcal{A}} \dots$, then it would correspond to an infinite chain $\mathcal{A}(\beta)(s_1) \succ \mathcal{A}(\beta)(s_2) \succ \dots$ (with β chosen arbitrarily). Thus $\succ_{\mathcal{A}}$ is well-founded. Irreflexivity and transitivity are proved similarly. □

Polynomial Orderings

Polynomial orderings:

Instance of the interpretation method:

The carrier set $U_{\mathcal{A}}$ is $\mathbb{N}$ or some subset of $\mathbb{N}$.

To every function symbol f with arity n we associate a polynomial $P_f(X_1, \dots, X_n) \in \mathbb{N}[X_1, \dots, X_n]$ with coefficients in $\mathbb{N}$ and indeterminates $X_1, \dots, X_n$. Then we define $f_{\mathcal{A}}(a_1, \dots, a_n) = P_f(a_1, \dots, a_n)$ for $a_i \in U_{\mathcal{A}}$.

Requirement 1:

If $a_1, \dots, a_n \in U_{\mathcal{A}}$, then $f_{\mathcal{A}}(a_1, \dots, a_n) \in U_{\mathcal{A}}$. (Otherwise, $\mathcal{A}$ would not be a Σ-algebra.)

Requirement 2:

$f_{\mathcal{A}}$ must be monotone (w. r. t. $\succ$).

From now on:

$U_{\mathcal{A}} = \{\, n \in \mathbb{N} \mid n \geq 1 \,\}$.

If $\mathrm{arity}(f) = 0$, then P_f is a constant ≥ 1.

If $\mathrm{arity}(f) = n \geq 1$, then P_f is a polynomial $P(X_1, \dots, X_n)$, such that every X_i occurs in some monomial with exponent at least 1 and non-zero coefficient.

$\Rightarrow$ Requirements 1 and 2 are satisfied.

The mapping from function symbols to polynomials can be extended to terms: A term t containing the variables $x_1, \dots, x_n$ yields a polynomial P_t with indeterminates $X_1, \dots, X_n$ (where X_i corresponds to $\beta(x_i)$).

Example:

$\Omega = \{b/0,\ f/1,\ g/3\}$
$P_b = 3, \quad P_f(X_1) = X_1^2, \quad P_g(X_1, X_2, X_3) = X_1 + X_2 X_3.$

Let $t = g(f(b), f(x), y)$, then $P_t(X, Y) = 9 + X^2 Y$.

If P, Q are polynomials in $\mathbb{N}[X_1, \dots, X_n]$, we write $P > Q$ if $P(a_1, \dots, a_n) > Q(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in U_{\mathcal{A}}$.

Clearly, $l \succ_{\mathcal{A}} r$ iff $P_l > P_r$ iff $P_l - P_r > 0$.

Question: Can we check $P_l - P_r > 0$ automatically?

Hilbert's 10th Problem:

Given a polynomial $P \in \mathbb{Z}[X_1, \dots, X_n]$ with integer coefficients, is $P = 0$ for some n-tuple of natural numbers?

Theorem 4.23 *Hilbert's 10th Problem is undecidable.*

Proposition 4.24 *Given a polynomial interpretation and two terms l, r, it is undecidable whether $P_l > P_r$.*

Proof. By reduction of Hilbert's 10th Problem. □

One easy case:

If we restrict to linear polynomials, deciding whether $P_l - P_r > 0$ is trivial:

$\sum k_i a_i + k > 0$ for all $a_1, \dots, a_n \geq 1$ if and only if

$k_i \geq 0$ for all $i \in \{1, \dots, n\}$,

and $\sum k_i + k > 0$

Another possible solution:

Test whether $P_l(a_1, \dots, a_n) > P_r(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in \{\, x \in \mathbb{R} \mid x \geq 1 \,\}$.

This is decidable (but hard). Since $U_{\mathcal{A}} \subseteq \{\, x \in \mathbb{R} \mid x \geq 1 \,\}$, it implies $P_l > P_r$.

Alternatively:

Use fast overapproximations.

Simplification Orderings

The *proper subterm ordering* $\rhd$ is defined by $s \rhd t$ if and only if $s|_p = t$ for some position $p \neq \varepsilon$ of s.

A rewrite ordering $\succ$ over $\mathrm{T}_\Sigma(X)$ is called *simplification ordering,* if it has the *subterm property:* $s \rhd t$ implies $s \succ t$ for all $s, t \in \mathrm{T}_\Sigma(X)$.

Example:

Let R_{emb} be the rewrite system $R_{\text{emb}} = \{\, f(x_1, \dots, x_n) \to x_i \mid f \in \Omega,\ 1 \leq i \leq n = \text{arity}(f) \,\}$.

Define $\rhd_{\text{emb}} = \to^{+}_{R_{\text{emb}}}$ and $\unrhd_{\text{emb}} = \to^{*}_{R_{\text{emb}}}$ *("homeomorphic embedding relation").*

$\rhd_{\text{emb}}$ is a simplification ordering.

Lemma 4.25 *If $\succ$ is a simplification ordering, then $s \rhd_{\text{emb}} t$ implies $s \succ t$ and $s \unrhd_{\text{emb}} t$ implies $s \succeq t$.*

Proof. Since $\succ$ is transitive and $\succeq$ is transitive and reflexive, it suffices to show that $s \to_{R_{\text{emb}}} t$ implies $s \succ t$. By definition, $s \to_{R_{\text{emb}}} t$ if and only if $s = s[l\sigma]$ and $t = s[r\sigma]$ for some rule $l \to r \in R_{\text{emb}}$. Obviously, $l \rhd r$ for all rules in R_{emb}, hence $l \succ r$. Since $\succ$ is a rewrite relation, $s = s[l\sigma] \succ s[r\sigma] = t$. □

Goal:

Show that every simplification ordering is well-founded (and therefore a reduction ordering).

Note: This works only for *finite* signatures!

To fix this for infinite signatures, the definition of simplification orderings and the definition of embedding have to be modified.

Theorem 4.26 ("Kruskal's Theorem") *Let Σ be a finite signature, let X be a finite set of variables. Then for every infinite sequence $t_1, t_2, t_3, \dots$ there are indices $j > i$ such that $t_j \unrhd_{\text{emb}} t_i$. ($\unrhd_{\text{emb}}$ is called* a well-partial-ordering (wpo).*)*

Proof. See Baader and Nipkow, page 113–115. □

Theorem 4.27 (Dershowitz) *If Σ is a finite signature, then every simplification ordering $\succ$ on $\mathrm{T}_\Sigma(X)$ is well-founded (and therefore a reduction ordering).*

Proof. Suppose that $t_1 \succ t_2 \succ t_3 \succ \dots$ is an infinite descending chain.

First assume that there is an $x \in \mathrm{var}(t_{i+1}) \setminus \mathrm{var}(t_i)$. Let $\sigma = \{x \mapsto t_i\}$, then $t_{i+1}\sigma \trianglerighteq x\sigma = t_i$ and therefore $t_i = t_i\sigma \succ t_{i+1}\sigma \succeq t_i$, contradicting reflexivity.

Consequently, $\mathrm{var}(t_i) \supseteq \mathrm{var}(t_{i+1})$ and $t_i \in \mathrm{T}_\Sigma(V)$ for all i, where V is the finite set $\mathrm{var}(t_1)$. By Kruskal's Theorem, there are $i < j$ with $t_i \trianglelefteq_{\mathrm{emb}} t_j$. Hence $t_i \preceq t_j$, contradicting $t_i \succ t_j$. □

There are reduction orderings that are not simplification orderings and terminating TRSs that are not contained in any simplification ordering.

Example:

Let $R = \{f(f(x)) \to f(g(f(x)))\}$.

R terminates and $\to_R^+$ is therefore a reduction ordering.

Assume that $\to_R$ were contained in a simplification ordering $\succ$. Then $f(f(x)) \to_R f(g(f(x)))$ implies $f(f(x)) \succ f(g(f(x)))$, and $f(g(f(x))) \trianglerighteq_{\mathrm{emb}} f(f(x))$ implies $f(g(f(x))) \succeq f(f(x))$, hence $f(f(x)) \succ f(f(x))$.

Path Orderings

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω.

The *lexicographic path ordering* $\succ_{\mathrm{lpo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{\mathrm{lpo}} t$ iff

(1) $t \in \mathrm{var}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and

  (a) $s_i \succeq_{\mathrm{lpo}} t$ for some i, or

  (b) $f \succ g$ and $s \succ_{\mathrm{lpo}} t_j$ for all j, or

  (c) $f = g$, $s \succ_{\mathrm{lpo}} t_j$ for all j, and $(s_1, \dots, s_m)\ (\succ_{\mathrm{lpo}})_{\mathrm{lex}}\ (t_1, \dots, t_n)$.

Lemma 4.28 $s \succ_{\mathrm{lpo}} t$ *implies* $\mathrm{var}(s) \supseteq \mathrm{var}(t)$.

Proof. By induction on $|s| + |t|$ and case analysis. □

Theorem 4.29 $\succ_{\text{lpo}}$ *is a simplification ordering on* $\mathrm{T}_\Sigma(X)$.

Proof. Show transitivity, subterm property, stability under substitutions, compatibility with Σ-operations, and irreflexivity, usually by induction on the sum of the term sizes and case analysis. Details: Baader and Nipkow, page 119/120. □

Theorem 4.30 *If the precedence* $\succ$ *is total, then the lexicographic path ordering* $\succ_{\text{lpo}}$ *is total on ground terms, i. e., for all* $s, t \in \mathrm{T}_\Sigma(\emptyset)$: $s \succ_{\text{lpo}} t \vee t \succ_{\text{lpo}} s \vee s = t$.

Proof. By induction on $|s| + |t|$ and case analysis. □

Recapitulation:

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω. The *lexicographic path ordering* $\succ_{\text{lpo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{\text{lpo}} t$ iff

(1) $t \in \mathrm{var}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and

- (a) $s_i \succeq_{\text{lpo}} t$ for some i, or
- (b) $f \succ g$ and $s \succ_{\text{lpo}} t_j$ for all j, or
- (c) $f = g$, $s \succ_{\text{lpo}} t_j$ for all j, and $(s_1, \dots, s_m)\ (\succ_{\text{lpo}})_{\text{lex}}\ (t_1, \dots, t_n)$.

There are several possibilities to compare subterms in (2)(c):

- compare list of subterms lexicographically left-to-right (*"lexicographic path ordering (lpo)"*, Kamin and Lévy)
- compare list of subterms lexicographically right-to-left (or according to some permutation π)
- compare multiset of subterms using the multiset extension (*"multiset path ordering (mpo)"*, Dershowitz)
- to each function symbol f with arity$(n) \geq 1$ associate a status $\in \{mul\} \cup \{\, lex_\pi \mid \pi : \{1, \dots, n\} \to \{1, \dots, n\}\,\}$ and compare according to that status (*"recursive path ordering (rpo) with status"*)

The Knuth-Bendix Ordering

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω, let $w : \Omega \cup X \to \mathbb{R}_0^+$ be a *weight function*, such that the following admissibility conditions are satisfied:

$w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in X$; $w(c) \geq w_0$ for all constants $c \in \Omega$.

If $w(f) = 0$ for some $f \in \Omega$ with $\text{arity}(f) = 1$, then $f \succeq g$ for all $g \in \Omega$.

The weight function w can be extended to terms recursively:

$$w(f(t_1, \dots, t_n)) = w(f) + \sum_{1 \leq i \leq n} w(t_i)$$

or alternatively

$$w(t) = \sum_{x \in \text{var}(t)} w(x) \cdot \#(x, t) + \sum_{f \in \Omega} w(f) \cdot \#(f, t).$$

where $\#(a, t)$ is the number of occurrences of a in t.

The *Knuth-Bendix ordering* $\succ_{\text{kbo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ and w is defined by: $s \succ_{\text{kbo}} t$ iff

(1) $\#(x, s) \geq \#(x, t)$ for all variables x and $w(s) > w(t)$, or

(2) $\#(x, s) \geq \#(x, t)$ for all variables x, $w(s) = w(t)$, and

  (a) $t = x$, $s = f^n(x)$ for some $n \geq 1$, or

  (b) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and $f \succ g$, or

  (c) $s = f(s_1, \dots, s_m)$, $t = f(t_1, \dots, t_m)$, and $(s_1, \dots, s_m)\ (\succ_{\text{kbo}})_{\text{lex}}\ (t_1, \dots, t_m)$.

Theorem 4.31 *The Knuth-Bendix ordering induced by $\succ$ and w is a simplification ordering on $\mathrm{T}_\Sigma(X)$.*

Proof. Baader and Nipkow, pages 125–129. □

Remark

If $\Pi \neq \emptyset$, then all the term orderings described in this section can also be used to compare non-equational atoms by treating predicate symbols like function symbols.