Superposition: Refutational Completeness

Construction of candidate interpretations
(Bachmair & Ganzinger 1990):

Let N be a set of clauses not containing $\bot$.
Using induction on the clause ordering we define sets of rewrite rules E_C and R_C for all $C \in G_\Sigma(N)$ as follows:

Assume that E_D has already been defined for all $D \in G_\Sigma(N)$ with $D \prec_C C$. Then $R_C = \bigcup_{D \prec_C C} E_D$.

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The set E_C contains the rewrite rule $s \to t$, if

(a) $C = C' \vee s \approx t$.

(b) $s \approx t$ is strictly maximal in C.

(c) $s \succ t$.

(d) C is false in R_C.

(e) C' is false in $R_C \cup \{s \to t\}$.

(f) s is irreducible w. r. t. R_C.

(g) no negative literal is selected in C'

In this case, C is called productive. Otherwise $E_C = \emptyset$.

Finally, $R_\infty = \bigcup_{D \in G_\Sigma(N)} E_D$.

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Lemma 6.5:

If $E_C = \{s \to t\}$ and $E_D = \{u \to v\}$, then $s \succ u$ if and only if $C \succ_C D$.

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Corollary 6.6:
The rewrite systems R_C and R_∞ are convergent.

Proof:
Obviously, $s \succ t$ for all rules $s \to t$ in R_C and R_∞.

Furthermore, it is easy to check that there are no critical pairs between any two rules: Assume that there are rules $u \to v$ in E_D and $s \to t$ in E_C such that u is a subterm of s. As $\succ$ is a reduction ordering that is total on ground terms, we get $u \prec s$ and therefore $D \prec_C C$ and $E_D \subseteq R_C$. But then s would be reducible by R_C, contradicting condition (f). $\square$

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Lemma 6.7:
If $D \preceq_C C$ and $E_C = \{s \rightarrow t\}$, then $s \succ u$ for every term u occurring in a negative literal in D and $s \succeq v$ for every term v occurring in a positive literal in D.

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Corollary 6.8:
If $D \in G_\Sigma(N)$ is true in R_D, then D is true in R_∞ and R_C for all $C \succ_C D$.

Proof:
If a positive literal of D is true in R_D, then this is obvious.

Otherwise, some negative literal $s \not\approx t$ of D must be true in R_D, hence $s \not\downarrow_{R_D} t$. As the rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than s and t, they cannot be used in a rewrite proof of $s \downarrow t$, hence $s \not\downarrow_{R_C} t$ and $s \not\downarrow_{R_\infty} t$. $\square$

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Corollary 6.9:
If $D = D' \vee u \approx v$ is productive, then D' is false and D is true in R_∞ and R_C for all $C \succ_C D$.

Proof:
Obviously, D is true in R_∞ and R_C for all $C \succ_C D$.

Since all negative literals of D' are false in R_D, it is clear that they are false in R_∞ and R_C. For the positive literals $u' \approx v'$ of D', condition (e) ensures that they are false in $R_D \cup \{u \to v\}$. Since $u' \preceq u$ and $v' \preceq u$ and all rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than u, these rules cannot be used in a rewrite proof of $u' \downarrow v'$, hence $u' \not\downarrow_{R_C} v'$ and $u' \not\downarrow_{R_\infty} v'$. $\square$

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Lemma 6.10 ("Lifting Lemma"):
Let C be a clause and let θ be a substitution such that $C\theta$ is ground. Then every equality resolution or equality factoring inference from $C\theta$ is a ground instance of an inference from C.

Proof:
Exercise. $\square$

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Lemma 6.11 ("Lifting Lemma"):
Let $D = D' \vee u \approx v$ and $C = C' \vee [\neg]\, s \approx t$ be two clauses (without common variables) and let θ be a substitution such that $D\theta$ and $C\theta$ are ground.

If there is a superposition inference between $D\theta$ and $C\theta$ where $u\theta$ and some subterm of $s\theta$ are overlapped, and $u\theta$ does not occur in $s\theta$ at or below a variable position of s, then the inference is a ground instance of a superposition inference from D and C.

Proof:
Exercise. $\square$

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Theorem 6.12 ("Model Construction"):
Let N be a set of clauses that is saturated up to redundancy and does not contain the empty clause. Then we have for every ground clause $C\theta \in G_\Sigma(N)$:

(i) $E_{C\theta} = \emptyset$ if and only if $C\theta$ is true in $R_{C\theta}$.

(ii) If $C\theta$ is redundant w. r. t. $G_\Sigma(N)$, then it is true in $R_{C\theta}$.

(iii) $C\theta$ is true in R_∞ and in R_D for every $D \in G_\Sigma(N)$ with $D \succ_C C\theta$.

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A Σ-interpretation $\mathcal{A}$ is called term-generated, if for every $b \in U_{\mathcal{A}}$ there is a ground term $t \in \mathsf{T}_{\Sigma}(\emptyset)$ such that $b = \mathcal{A}(\beta)(t)$.

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Lemma 6.13:

Let N be a set of (universally quantified) Σ-clauses and let $\mathcal{A}$ be a term-generated Σ-interpretation. Then $\mathcal{A}$ is a model of $G_\Sigma(N)$ if and only if it is a model of N.

Proof:

($\Rightarrow$): Let $\mathcal{A} \models G_\Sigma(N)$; let $(\forall \vec{x} C) \in N$.
Then $\mathcal{A} \models \forall \vec{x} C$ iff $\mathcal{A}(\gamma[x_i \mapsto a_i])(C) = 1$ for all γ and a_i.
Choose ground terms t_i such that $\mathcal{A}(\gamma)(t_i) = a_i$; define θ such that $x_i\theta = t_i$, then $\mathcal{A}(\gamma[x_i \mapsto a_i])(C) = \mathcal{A}(\gamma \circ \theta)(C) = \mathcal{A}(\gamma)(C\theta) = 1$ since $C\theta \in G_\Sigma(N)$.

($\Leftarrow$): Let $\mathcal{A}$ be a model of N; let $C \in N$ and $C\theta \in G_\Sigma(N)$.
Then $\mathcal{A}(\gamma)(C\theta) = \mathcal{A}(\gamma \circ \theta)(C) = 1$ since $\mathcal{A} \models N$. $\square$

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Theorem 6.14 (Refutational Completeness: Static View):
Let N be a set of clauses that is saturated up to redundancy. Then N has a model if and only if N does not contain the empty clause.

Proof:
If $\bot \in N$, then obviously N does not have a model.
If $\bot \notin N$, then the interpretation R_∞ (that is, $\mathrm{T}_\Sigma(\emptyset)/R_\infty$) is a model of all ground instances in $G_\Sigma(N)$ according to part (iii) of the model construction theorem.
As $\mathrm{T}_\Sigma(\emptyset)/R_\infty$ is term generated, it is a model of N. $\square$

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So far, we have considered only inference rules that add new clauses to the current set of clauses (corresponding to the *Deduce* rule of Knuth-Bendix Completion).

In other words, we have derivations of the form $N_0 \vdash N_1 \vdash N_2 \vdash \ldots$, where each N_{i+1} is obtained from N_i by adding the consequence of some inference from clauses in N_i.

Under which circumstances are we allowed to delete (or simplify) a clause during the derivation?

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A run of the superposition calculus is a sequence
$N_0 \vdash N_1 \vdash N_2 \vdash \dots$, such that
(i) $N_i \models N_{i+1}$, and
(ii) all clauses in $N_i \setminus N_{i+1}$ are redundant w. r. t. N_{i+1}.

In other words, during a run we may add a new clause if it follows from the old ones, and we may delete a clause, if it is redundant w. r. t. the remaining ones.

For a run, $N_\infty = \bigcup_{i \geq 0} N_i$ and $N_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} N_j$.
The set N_* of all persistent clauses is called the limit of the run.

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Lemma 6.15:

If $N \subseteq N'$, then $Red(N) \subseteq Red(N')$.

Proof:

Obvious. $\square$

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Lemma 6.16:

If $N' \subseteq \mathit{Red}(N)$, then $\mathit{Red}(N) \subseteq \mathit{Red}(N \setminus N')$.

Proof:

Follows from the compactness of first-order logic and the well-foundedness of the multiset extension of the clause ordering.

$\square$

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Lemma 6.17:

Let $N_0 \vdash N_1 \vdash N_2 \vdash \ldots$ be a run.

Then $\mathit{Red}(N_i) \subseteq \mathit{Red}(N_\infty)$ and $\mathit{Red}(N_i) \subseteq \mathit{Red}(N_*)$ for every i.

Proof:

Exercise. $\square$

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Corollary 6.18:
$N_i \subseteq N_* \cup \mathit{Red}(N_*)$ for every i.

Proof:
If $C \in N_i \setminus N_*$, then there is a $k \geq i$ such that $C \in N_k \setminus N_{k+1}$, so C must be redundant w. r. t. N_{k+1}.
Consequently, C is redundant w. r. t. N_*. $\square$

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A run is called fair, if the conclusion of every inference from clauses in $N_* \setminus \mathit{Red}(N_*)$ is contained in some $N_i \cup \mathit{Red}(N_i)$.

Lemma 6.19:

If a run is fair, then its limit is saturated up to redundancy.

Proof:

If the run is fair, then the conclusion of every inference from non-redundant clauses in N_* is contained in some $N_i \cup \mathit{Red}(N_i)$, and therefore contained in $N_* \cup \mathit{Red}(N_*)$.

Hence N_* is saturated up to redundancy. $\square$

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Theorem 6.20 (Refutational Completeness: Dynamic View):
Let $N_0 \vdash N_1 \vdash N_2 \vdash \ldots$ be a fair run, let N_* be its limit.
Then N_0 has a model if and only if $\bot \notin N_*$.

Proof:
($\Leftarrow$): By fairness, N_* is saturated up to redundancy.
If $\bot \notin N_*$, then it has a term-generated model.
Since every clause in N_0 is contained in N_* or redundant w. r. t. N_*, this model is also a model of $G_\Sigma(N_0)$ and therefore a model of N_0.

($\Rightarrow$): Obvious, since $N_0 \models N_*$. $\square$