Saturation of Sets of General Clauses

Corollary 3.27:

Let N be a set of general clauses saturated under Res, i. e., $\mathit{Res}(N) \subseteq N$. Then also $G_\Sigma(N)$ is saturated, that is,

$$\mathit{Res}(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Saturation of Sets of General Clauses

Proof:

W.l.o.g. we may assume that clauses in N are pairwise variable-disjoint. (Otherwise make them disjoint, and this renaming process changes neither $\mathit{Res}(N)$ nor $G_\Sigma(N)$.)

Let $C' \in \mathit{Res}(G_\Sigma(N))$, meaning (i) there exist resolvable ground instances $D\sigma$ and $C\rho$ of N with resolvent C', or else (ii) C' is a factor of a ground instance $C\sigma$ of C.

Case (i): By the Lifting Lemma, D and C are resolvable with a resolvent C'' with $C''\tau = C'$, for a suitable substitution τ. As $C'' \in N$ by assumption, we obtain that $C' \in G_\Sigma(N)$.

Case (ii): Similar. $\square$

Herbrand's Theorem

Lemma 3.28:

Let N be a set of Σ-clauses, let $\mathcal{A}$ be an interpretation. Then $\mathcal{A} \models N$ implies $\mathcal{A} \models G_\Sigma(N)$.

Lemma 3.29:

Let N be a set of Σ-clauses, let $\mathcal{A}$ be a *Herbrand* interpretation. Then $\mathcal{A} \models G_\Sigma(N)$ implies $\mathcal{A} \models N$.

Herbrand's Theorem

Theorem 3.30 (Herbrand):
A set N of Σ-clauses is satisfiable if and only if it has a Herbrand model over Σ.

Proof:
The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let $N \not\models \bot$.

$$
\begin{aligned}
N \not\models \bot &\Rightarrow \bot \notin \mathit{Res}^*(N) && \text{(resolution is sound)} \\
&\Rightarrow \bot \notin G_\Sigma(\mathit{Res}^*(N)) \\
&\Rightarrow G_\Sigma(\mathit{Res}^*(N))_{\mathcal{I}} \models G_\Sigma(\mathit{Res}^*(N)) && \text{(Thm. 3.17; Cor. 3.27)} \\
&\Rightarrow G_\Sigma(\mathit{Res}^*(N))_{\mathcal{I}} \models \mathit{Res}^*(N) && \text{(Lemma 3.29)} \\
&\Rightarrow G_\Sigma(\mathit{Res}^*(N))_{\mathcal{I}} \models N && (N \subseteq \mathit{Res}^*(N)) \quad \Box
\end{aligned}
$$

The Theorem of Löwenheim-Skolem

Theorem 3.31 (Löwenheim–Skolem):
Let Σ be a countable signature and let S be a set of closed Σ-formulas. Then S is satisfiable iff S has a model over a countable universe.

Proof:
If both X and Σ are countable, then S can be at most countably infinite. Now generate, maintaining satisfiability, a set N of clauses from S. This extends Σ by at most countably many new Skolem functions to Σ'. As Σ' is countable, so is $T_{\Sigma'}$, the universe of Herbrand-interpretations over Σ'. Now apply Theorem 3.30. $\square$

Refutational Completeness of General Resolution

Theorem 3.32:

Let N be a set of general clauses where $Res(N) \subseteq N$. Then

$$N \models \bot \Leftrightarrow \bot \in N.$$

Proof:

Let $Res(N) \subseteq N$. By Corollary 3.27: $Res(G_\Sigma(N)) \subseteq G_\Sigma(N)$

$N \models \bot \Leftrightarrow G_\Sigma(N) \models \bot$ (Lemma 3.28/3.29; Theorem 3.30)

$\Leftrightarrow \bot \in G_\Sigma(N)$ (propositional resolution sound and complete)

$\Leftrightarrow \bot \in N$ $\square$

Compactness of Predicate Logic

Theorem 3.33 (Compactness Theorem for First-Order Logic):
Let S be a set of first-order formulas. S is unsatisfiable iff some finite subset $S' \subseteq S$ is unsatisfiable.

Proof:
The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let S be unsatisfiable and let N be the set of clauses obtained by Skolemization and CNF transformation of the formulas in S. Clearly $Res^*(N)$ is unsatisfiable. By Theorem 3.32, $\bot \in Res^*(N)$, and therefore $\bot \in Res^n(N)$ for some $n \in \mathbb{N}$. Consequently, $\bot$ has a finite resolution proof B of depth $\leq n$. Choose S' as the subset of formulas in S such that the corresponding clauses contain the assumptions (leaves) of B. $\Box$

3.11 First-Order Superposition with Selection

Motivation: Search space for *Res* *very* large.

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem 2.13) one only needs to resolve and factor maximal atoms
   $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
   $\Rightarrow$ *ordering restrictions*

2. In the proof, it does not really matter with which negative literal an inference is performed
   $\Rightarrow$ choose a negative literal don't-care-nondeterministically
   $\Rightarrow$ *selection*

Selection Functions

A selection function is a mapping

$$\text{sel} : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \vee \neg A \vee B$$

$$\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$$

Selection Functions

Intuition:

- If a clause has at least one selected literal, compute only inferences that involve a selected literal.
- If a clause has no selected literals, compute only inferences that involve a maximal literal.

Orderings for Terms, Atoms, Clauses

For first-order logic an ordering on the signature symbols is not sufficient to compare atoms, e.g., how to compare $P(a)$ and $P(b)$?

We propose the Knuth-Bendix Ordering for terms, atoms (with variables) which is then lifted as in the propositional case to literals and clauses.

The Knuth-Bendix Ordering (Simple)

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a total ordering ("precedence") on $\Omega \cup \Pi$, let $w : \Omega \cup \Pi \cup X \to \mathbb{R}^+$ be a weight function, satisfying $w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in X$ and $w(c) \geq w_0$ for all constants $c \in \Omega$.

The weight function w can be extended to terms (atoms) as follows:

$$w(f(t_1, \ldots, t_n)) = w(f) + \sum_{1 \leq i \leq n} w(t_i)$$

$$w(P(t_1, \ldots, t_n)) = w(P) + \sum_{1 \leq i \leq n} w(t_i)$$

The Knuth-Bendix Ordering (Simple)

The Knuth-Bendix ordering $\succ_{\text{kbo}}$ on $\mathrm{T}_\Sigma(X)$ (atoms) induced by $\succ$ and w is defined by: $s \succ_{\text{kbo}} t$ iff

(1) $\#(x, s) \geq \#(x, t)$ for all variables x and $w(s) > w(t)$, or

(2) $\#(x, s) \geq \#(x, t)$ for all variables x, $w(s) = w(t)$, and

(a) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f \succ g$, or

(b) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and $(s_1, \ldots, s_m)$ $(\succ_{\text{kbo}})_{\text{lex}}$ $(t_1, \ldots, t_m)$.

where $\#(s, t) = |\{p \mid t|_p = s\}|$.

The Knuth-Bendix Ordering (Simple)

Proposition 3.34:

The Knuth-Bendix ordering $\succ_{\mathrm{kbo}}$ is

(1) a strict partial well-founded ordering on terms (atoms).

(2) stable under substitution: if $s \succ_{\mathrm{kbo}} t$ then $s\sigma \succ_{\mathrm{kbo}} t\sigma$ for any σ.

(3) total on ground terms (ground atoms).

Superposition Calculus $\mathit{Sup}^{\succ}_{\mathsf{sel}}$

The resolution calculus $\mathit{Sup}^{\succ}_{\mathsf{sel}}$ is parameterized by

- a selection function sel
- and a total and well-founded atom ordering $\succ$.

Superposition Calculus $Sup^{\succ}_{\mathrm{sel}}$

In the completeness proof, we talk about (strictly) maximal literals of *ground* clauses.

In the non-ground calculus, we have to consider those literals that correspond to (strictly) maximal literals of ground instances:

A literal L is called [strictly] maximal in a clause C if and only if there exists a ground substitution σ such that $L\sigma$ is [strictly] maximal in $C\sigma$ (i.e., if for no other L' in C: $L\sigma \prec L'\sigma$ [$L\sigma \preceq L'\sigma$]).

Superposition Calculus $Sup^{\succ}_{\text{sel}}$

$$\frac{D \vee B \qquad C \vee \neg A}{(D \vee C)\sigma}$$ [Superposition Left with Selection]

if the following conditions are satisfied:

(i) $\sigma = \text{mgu}(A, B)$;

(ii) $B\sigma$ strictly maximal in $D\sigma \vee B\sigma$;

(iii) nothing is selected in $D \vee B$ by sel;

(iv) either $\neg A$ is selected, or else nothing is selected in $C \vee \neg A$ and $\neg A\sigma$ is maximal in $C\sigma \vee \neg A\sigma$.

Superposition Calculus $Sup^{\succ}_{\text{sel}}$

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \qquad \text{[Factoring]}$$

if the following conditions are satisfied:

(i) $\sigma = \text{mgu}(A, B)$;

(ii) $A\sigma$ is maximal in $C\sigma \vee A\sigma \vee B\sigma$;

(iii) nothing is selected in $C \vee A \vee B$ by sel.

Special Case: Propositional Logic

For ground clauses the superposition inference rule simplifies to

$$\frac{D \vee P \qquad C \vee \neg P}{D \vee C}$$

if the following conditions are satisfied:

(i) $P \succ D$;

(ii) nothing is selected in $D \vee P$ by sel;

(iii) $\neg P$ is selected in $C \vee \neg P$, or else nothing is selected in $C \vee \neg P$ and $\neg P \succeq \max(C)$.

Note: For positive literals, $P \succ D$ is the same as $P \succ \max(D)$.

Special Case: Propositional Logic

Analogously, the factoring rule simplifies to

$$\frac{C \vee P \vee P}{C \vee P}$$

if the following conditions are satisfied:

(i) P is the largest literal in $C \vee P \vee P$;

(ii) nothing is selected in $C \vee P \vee P$ by sel.

Search Spaces Become Smaller

1	$P \vee Q$	
2	$P \vee \boxed{\neg Q}$	
3	$\neg P \vee Q$	
4	$\neg P \vee \boxed{\neg Q}$	
5	$Q \vee Q$	Res 1, 3
6	Q	Fact 5
7	$\neg P$	Res 6, 4
8	P	Res 6, 2
9	$\perp$	Res 8, 7

we assume $P \succ Q$ and sel as indicated by $\boxed{X}$. The maximal literal in a clause is depicted in red.

With this ordering and selection function the refutation proceeds strictly deterministically in this example. Generally, proof search will still be non-deterministic but the search space will be much smaller than with unrestricted resolution.

Avoiding Rotation Redundancy

From

$$\dfrac{\dfrac{C_1 \vee P \qquad C_2 \vee \neg P \vee Q}{C_1 \vee C_2 \vee Q} \qquad C_3 \vee \neg Q}{C_1 \vee C_2 \vee C_3}$$

we can obtain by rotation

$$\dfrac{C_1 \vee P \qquad \dfrac{C_2 \vee \neg P \vee Q \qquad C_3 \vee \neg Q}{C_2 \vee \neg P \vee C_3}}{C_1 \vee C_2 \vee C_3}$$

another proof of the same clause. In large proofs many rotations are possible. However, if $P \succ Q$, then the second proof does not fulfill the orderings restrictions.

Avoiding Rotation Redundancy

Conclusion: In the presence of orderings restrictions (however one chooses $\succ$) no rotations are possible. In other words, orderings identify exactly one representant in any class of rotation-equivalent proofs.

Lifting Lemma for $Sup^{\succ}_{\mathrm{sel}}$

Lemma 3.35:

Let D and C be variable-disjoint clauses. If

$$\begin{array}{ccc} D & & C \\ \downarrow \sigma & & \downarrow \rho \\ \end{array}$$

$$\frac{D\sigma \qquad\qquad C\rho}{C'} \qquad [\text{propositional inference in } Sup^{\succ}_{\mathrm{sel}}]$$

and if $\mathrm{sel}(D\sigma) \simeq \mathrm{sel}(D)$, $\mathrm{sel}(C\rho) \simeq \mathrm{sel}(C)$ (that is, "corresponding" literals are selected), then there exists a substitution τ such that

$$\frac{D \qquad\qquad C}{C''} \qquad [\text{inference in } Sup^{\succ}_{\mathrm{sel}}]$$

$$\downarrow \tau$$

$$C' = C''\tau$$

Lifting Lemma for $Sup^{\succ}_{\mathrm{sel}}$

An analogous lifting lemma holds for factorization.

Saturation of General Clause Sets

Corollary 3.36:

Let N be a set of general clauses saturated under $Sup^{\succ}_{\text{sel}}$, i. e., $Sup^{\succ}_{\text{sel}}(N) \subseteq N$. Then there exists a selection function sel′ such that $\text{sel}\,|_N = \text{sel}'\,|_N$ and $G_\Sigma(N)$ is also saturated, i. e.,

$$Sup^{\succ}_{\text{sel}'}(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Proof:

We first define the selection function sel′ such that $\text{sel}'(C) = \text{sel}(C)$ for all clauses $C \in G_\Sigma(N) \cap N$. For $C \in G_\Sigma(N) \setminus N$ we choose a fixed but arbitrary clause $D \in N$ with $C \in G_\Sigma(D)$ and define $\text{sel}'(C)$ to be those occurrences of literals that are ground instances of the occurrences selected by sel in D. Then proceed as in the proof of Cor. 3.27 using the above lifting lemma. □

Soundness and Refutational Completeness

Theorem 3.37:

Let $\succ$ be an atom ordering and sel a selection function such that $Sup^{\succ}_{\mathrm{sel}}(N) \subseteq N$. Then

$$N \models \bot \Leftrightarrow \bot \in N$$

Proof:

The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part consider the propositional level: Construct a candidate interpretation $N_{\mathcal{I}}$ as for superposition without selection, except that clauses C in N that have selected literals are not productive, even when they are false in N_C and when their maximal atom occurs only once and positively. The result then follows by Corollary 3.36. $\square$

Craig-Interpolation

A theoretical application of superposition is Craig-Interpolation:

Theorem 3.38 (Craig 1957):
Let ϕ and ψ be two propositional formulas such that $\phi \models \psi$.
Then there exists a formula χ (called the interpolant for $\phi \models \psi$), such that χ contains only prop. variables occurring both in ϕ and in ψ, and such that $\phi \models \chi$ and $\chi \models \psi$.

Craig-Interpolation

Proof:

Translate ϕ and $\neg\psi$ into CNF. let N and M, resp., denote the resulting clause set. Choose an atom ordering $\succ$ for which the prop. variables that occur in ϕ but not in ψ are maximal. Saturate N into N^* w. r. t. $\mathit{Sup}^{\succ}_{\mathrm{sel}}$ with an empty selection function sel . Then saturate $N^* \cup M$ w. r. t. $\mathit{Sup}^{\succ}_{\mathrm{sel}}$ to derive $\bot$. As N^* is already saturated, due to the ordering restrictions only inferences need to be considered where premises, if they are from N^*, only contain symbols that also occur in ψ. The conjunction of these premises is an interpolant χ. The theorem also holds for first-order formulas. For universal formulas the above proof can be easily extended. In the general case, a proof based on superposition technology is more complicated because of Skolemization. $\square$