Confluence

Let $(A, \rightarrow)$ be a rewrite system.

b and $c \in A$ are joinable, if there is an a such that $b \rightarrow^* a \; {}^*\!\!\leftarrow c$.
Notation: $b \downarrow c$.

The relation $\rightarrow$ is called

Church-Rosser, if $b \leftrightarrow^* c$ implies $b \downarrow c$.

confluent, if $b \; {}^*\!\!\leftarrow a \rightarrow^* c$ implies $b \downarrow c$.

locally confluent, if $b \leftarrow a \rightarrow c$ implies $b \downarrow c$.

convergent, if it is confluent and terminating.

Confluence

For a rewrite system $(M, \to)$ consider a sequence of elements a_i that are pairwise connected by the symmetric closure, i.e., $a_1 \leftrightarrow a_2 \leftrightarrow a_3 \ldots \leftrightarrow a_n$. We say that a_i is a peak in such a sequence, if actually $a_{i-1} \leftarrow a_i \to a_{i+1}$.

Confluence

Theorem 1.11:

The following properties are equivalent:

(i) $\rightarrow$ has the Church-Rosser property.

(ii) $\rightarrow$ is confluent.

Confluence

Lemma 1.12:

If $\rightarrow$ is confluent, then every element has at most one normal form.

Corollary 1.13:

If $\rightarrow$ is normalizing and confluent, then every element b has a unique normal form.

Proposition 1.14:

If $\rightarrow$ is normalizing and confluent, then $b \leftrightarrow^* c$ if and only if $b{\downarrow} = c{\downarrow}$.

Confluence and Local Confluence

Theorem 1.15 ("Newman's Lemma"):
If a terminating relation $\rightarrow$ is locally confluent, then it is confluent.

Part 2: Propositional Logic

Propositional logic

- logic of truth values
- decidable (but **NP**-complete)
- can be used to describe functions over a finite domain
- industry standard for many analysis/verification tasks
- growing importance for discrete optimization problems (Automated Reasoning II)

2.1 Syntax

- propositional variables
- logical connectives
  $\Rightarrow$ Boolean connectives and constants

Propositional Variables

Let Σ be a set of propositional variables also called the signature of the (propositional) logic.

We use letters P, Q, R, S, to denote propositional variables.

Propositional Formulas

PROP(Σ) is the set of propositional formulas over Σ inductively defined as follows:

ϕ, ψ	$::=$	$\bot$	(falsum)
	$\mid$	$\top$	(verum)
	$\mid$	$P, \quad P \in \Sigma$	(atomic formula)
	$\mid$	$\neg\phi$	(negation)
	$\mid$	$(\phi \wedge \psi)$	(conjunction)
	$\mid$	$(\phi \vee \psi)$	(disjunction)
	$\mid$	$(\phi \rightarrow \psi)$	(implication)
	$\mid$	$(\phi \leftrightarrow \psi)$	(equivalence)

Notational Conventions

As a notational convention we assume that $\neg$ binds strongest, so $\neg P \lor Q$ is actually a shorthand for $(\neg P) \lor Q$. For all other logical connectives we will explicitly put parenthesis when needed. From the semantics we will see that $\land$ and $\lor$ are associative and commutative. Therefore instead of $((P \land Q) \land R)$ we simply write $P \land Q \land R$.

Automated reasoning is very much formula manipulation. In order to precisely represent the manipulation of a formula, we introduce positions.

Formula Manipulation

A position is a word over $\mathbb{N}$. The set of positions of a formula ϕ is inductively defined by

$$
\begin{aligned}
\mathrm{pos}(\phi) &:= \{\epsilon\} \text{ if } \phi \in \{\top, \bot\} \text{ or } \phi \in \Sigma \\
\mathrm{pos}(\neg\phi) &:= \{\epsilon\} \cup \{1p \mid p \in \mathrm{pos}(\phi)\} \\
\mathrm{pos}(\phi \circ \psi) &:= \{\epsilon\} \cup \{1p \mid p \in pos(\phi)\} \cup \{2p \mid p \in \mathrm{pos}(\psi)\}
\end{aligned}
$$

where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Formula Manipulation

The prefix order $\leq$ on positions is defined by $p \leq q$ if there is some p' such that $pp' = q$.

Note that the prefix order is partial, e.g., the positions 12 and 21 are not comparable, they are "parallel", see below.

By $<$ we denote the strict part of $\leq$, i.e., $p < q$ if $p \leq q$ but not $q \leq p$. By $\parallel$ we denote incomparable positions, i.e., $p \parallel q$ if neither $p \leq q$, nor $q \leq p$. Then we say that p is above q if $p \leq q$, p is strictly above q if $p < q$, and p and q are parallel if $p \parallel q$.

Formula Manipulation

The size of a formula ϕ is given by the cardinality of $\text{pos}(\phi)$: $|\phi| := |\text{pos}(\phi)|$.

The subformula of ϕ at position $p \in \text{pos}(\phi)$ is recursively defined by $\phi|_\epsilon := \phi$ and $(\phi_1 \circ \phi_2)|_{ip} := \phi_i|_p$ where $i \in \{1, 2\}$, $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Formula Manipulation

Finally, the replacement of a subformula at position $p \in \mathrm{pos}(\phi)$ by a formula ψ is recursively defined by

$$
\begin{aligned}
\phi[\psi]_\epsilon &:= \psi \\
(\neg\phi)[\psi]_{1p} &:= \neg(\phi[\psi]_p) \\
(\phi_1 \circ \phi_2)[\psi]_{1p} &:= (\phi_1[\psi]_p \circ \phi_2) \\
(\phi_1 \circ \phi_2)[\psi]_{2p} &:= (\phi_1 \circ \phi_2[\psi]_p)
\end{aligned}
$$

where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Formula Manipulation

Example 2.1:
The set of positions for the formula $\phi = (A \wedge B) \rightarrow (A \vee B)$ is $\text{pos}(\phi) = \{\epsilon, 1, 11, 12, 2, 21, 22\}$. The subformula at position 22 is B, $\phi|_{22} = B$ and replacing this formula by $A \leftrightarrow B$ results in $\phi[A \leftrightarrow B]_{22} = (A \wedge B) \rightarrow (A \vee (A \leftrightarrow B))$.

Formula Manipulation

A further prerequisite for efficient formula manipulation is the polarity of a subformula ψ of ϕ. The polarity determines the number of "negations" starting from ϕ down to ψ. It is 1 for an even number along the path, -1 for an odd number and 0 if there is at least one equivalence connective along the path.

Formula Manipulation

The polarity of a subformula ψ of ϕ at position p, $i \in \{1, 2\}$ is recursively defined by

$$\begin{aligned}
\mathrm{pol}(\phi, \epsilon) &:= 1 \\
\mathrm{pol}(\neg\phi, 1p) &:= -\,\mathrm{pol}(\phi, p) \\
\mathrm{pol}(\phi_1 \circ \phi_2, ip) &:= \mathrm{pol}(\phi_i, p) \text{ if } \circ \in \{\wedge, \vee\} \\
\mathrm{pol}(\phi_1 \rightarrow \phi_2, 1p) &:= -\,\mathrm{pol}(\phi_2, p) \\
\mathrm{pol}(\phi_1 \rightarrow \phi_2, 2p) &:= \mathrm{pol}(\phi_2, p) \\
\mathrm{pol}(\phi_1 \leftrightarrow \phi_2, ip) &:= 0
\end{aligned}$$

Formula Manipulation

Example 2.2:

We reuse the formula $\phi = (A \land B) \rightarrow (A \lor B)$ Then $\operatorname{pol}(\phi, 1) = \operatorname{pol}(\phi, 11) = -1$ and $\operatorname{pol}(\phi, 2) = \operatorname{pol}(\phi, 22) = 1$. For the formula $\phi' = (A \land B) \leftrightarrow (A \lor B)$ we get $\operatorname{pol}(\phi', \epsilon) = 1$ and $\operatorname{pol}(\phi', p) = 0$ for all other $p \in \operatorname{pos}(\phi')$, $p \neq \epsilon$.