

## 6.4 Superposition

Goal:

Combine the ideas of superposition for first-order logic without equality (overlap maximal literals in a clause) and Knuth-Bendix completion (overlap maximal sides of equations) to get a calculus for equational clauses.

### Observation

It is possible to encode an arbitrary predicate  $p$  using a function  $f_p$  and a new constant  $tt$ :

$$\begin{array}{lcl} P(t_1, \dots, t_n) & \rightsquigarrow & f_P(t_1, \dots, t_n) \approx tt \\ \neg P(t_1, \dots, t_n) & \rightsquigarrow & \neg f_P(t_1, \dots, t_n) \approx tt \end{array}$$

In equational logic it is therefore sufficient to consider the case that  $\Pi = \emptyset$ , i. e., equality is the only predicate symbol.

Abbreviation:  $s \not\approx t$  instead of  $\neg s \approx t$ .

### The Superposition Calculus – Informally

Conventions:

From now on:  $\Pi = \emptyset$  (equality is the only predicate).

Inference rules are to be read modulo symmetry of the equality symbol.

We will first explain the ideas and motivations behind the superposition calculus and its completeness proof. Precise definitions will be given later.

Ground inference rules:

$$\text{Superposition Right: } \frac{D' \vee t \approx t' \quad C' \vee s[t] \approx s'}{D' \vee C' \vee s[t'] \approx s'}$$

$$\text{Superposition Left: } \frac{D' \vee t \approx t' \quad C' \vee s[t] \not\approx s'}{D' \vee C' \vee s[t'] \not\approx s'}$$

$$\text{Equality Resolution: } \frac{C' \vee s \not\approx s}{C'}$$

(Note: We will need one further inference rule.)

Ordering restrictions:

Some considerations:

The literal ordering must depend primarily on the larger term of an equation.

As in the resolution case, negative literals must be a bit larger than the corresponding positive literals.

Additionally, we need the following property: If  $s \succ t \succ u$ , then  $s \not\approx u$  must be larger than  $s \approx t$ . In other words, we must compare first the larger term, then the polarity, and finally the smaller term.

The following construction has the required properties:

Let  $\succ$  be a *reduction ordering that is total on ground terms*.

To a positive literal  $s \approx t$ , we assign the multiset  $\{s, t\}$ , to a negative literal  $s \not\approx t$  the multiset  $\{s, s, t, t\}$ . The *literal ordering*  $\succ_L$  compares these multisets using the multiset extension of  $\succ$ .

The *clause ordering*  $\succ_C$  compares clauses by comparing their multisets of literals using the multiset extension of  $\succ_L$ .

Ordering restrictions:

Ground inferences are necessary only if the following conditions are satisfied:

- In superposition inferences, the left premise is smaller than the right premise.
- The literals that are involved in the inferences are maximal in the respective clauses (strictly maximal for positive literals in superposition inferences).
- In these literals, the lhs is greater than or equal to the rhs (in superposition inferences: greater than the rhs).

Model construction:

We want to use roughly the same ideas as in the completeness proof for superposition on first-order without equality.

But: a Herbrand interpretation does not work for equality: The equality symbol  $\approx$  must be interpreted by equality in the interpretation.

Solution: Define a set  $E$  of ground equations and take  $T_\Sigma(\emptyset)/E = T_\Sigma(\emptyset)/\approx_E$  as the universe.

Then two ground terms  $s$  and  $t$  are equal in the interpretation, if and only if  $s \approx_E t$ .

If  $E$  is a terminating and confluent rewrite system  $R$ , then two ground terms  $s$  and  $t$  are equal in the interpretation, if and only if  $s \downarrow_R t$ .

One problem:

In the completeness proof for the resolution calculus, the following property holds:

If  $C = C' \vee A$  with a strictly maximal and positive literal  $A$  is false in the current interpretation, then adding  $A$  to the current interpretation cannot make any literal of  $C'$  true.

This does not hold for superposition:

Let  $b \succ c \succ d$ . Assume that the current rewrite system (representing the current interpretation) contains the rule  $c \rightarrow d$ . Now consider the clause  $b \approx c \vee b \approx d$ .

We need a further inference rule to deal with clauses of this kind, either the “Merging Paramodulation” rule of Bachmair and Ganzinger or the following “Equality Factoring” rule due to Nieuwenhuis:

$$\text{Equality Factoring: } \frac{C' \vee s \approx t' \vee s \approx t}{C' \vee t \not\approx t' \vee s \approx t}$$

Note: This inference rule subsumes the usual factoring rule.

How do the non-ground versions of the inference rules for superposition look like?

Main idea as in non-equational first-order case:

Replace identity by unifiability. Apply the mgu to the resulting clause. In the ordering restrictions, replace  $\succ$  by  $\not\prec$ .

However:

As in Knuth-Bendix completion, we do not want to consider overlaps at or below a variable position.

Consequence: there are inferences between ground instances  $D\theta$  and  $C\theta$  of clauses  $D$  and  $C$  which are *not* ground instances of inferences between  $D$  and  $C$ .

Such inferences have to be treated in a special way in the completeness proof.

## The Superposition Calculus – Formally

Until now, we have seen most of the ideas behind the superposition calculus and its completeness proof.

We will now start again from the beginning giving precise definitions and proofs.

Inference rules are applied with respect to the commutativity of equality  $\approx$ .

Inference rules:

$$\text{Superposition Right: } \frac{D' \vee t \approx t' \quad C' \vee s[u] \approx s'}{(D' \vee C' \vee s[t'] \approx s')\sigma}$$

where  $\sigma = \text{mgu}(t, u)$  and  
 $u$  is not a variable.

$$\text{Superposition Left: } \frac{D' \vee t \approx t' \quad C' \vee s[u] \not\approx s'}{(D' \vee C' \vee s[t'] \not\approx s')\sigma}$$

where  $\sigma = \text{mgu}(t, u)$  and  
 $u$  is not a variable.

$$\text{Equality Resolution: } \frac{C' \vee s \not\approx s'}{C'\sigma}$$

where  $\sigma = \text{mgu}(s, s')$ .

$$\text{Equality Factoring: } \frac{C' \vee s' \approx t' \vee s \approx t}{(C' \vee t \not\approx t' \vee s \approx t')\sigma}$$

where  $\sigma = \text{mgu}(s, s')$ .

**Theorem 6.4** *All inference rules of the superposition calculus are correct, i. e., for every rule*

$$\frac{C_n, \dots, C_1}{C_0}$$

*we have  $\{C_1, \dots, C_n\} \models C_0$ .*

**Proof.** Exercise. □

Orderings:

Let  $\succ$  be a *reduction ordering that is total on ground terms*.

To a positive literal  $s \approx t$ , we assign the multiset  $\{s, t\}$ , to a negative literal  $s \not\approx t$  the multiset  $\{s, s, t, t\}$ . The *literal ordering*  $\succ_L$  compares these multisets using the multiset extension of  $\succ$ .

The *clause ordering*  $\succ_C$  compares clauses by comparing their multisets of literals using the multiset extension of  $\succ_L$ .

Inferences have to be computed only if the following ordering restrictions are satisfied:

- In superposition inferences, after applying the unifier to both premises, the left premise is not greater than or equal to the right one.
- The last literal in each premise is maximal in the respective premise, i. e., there exists no greater literal (strictly maximal for positive literals in superposition inferences, i. e., there exists no greater or equal literal).
- In these literals, the lhs is not smaller than the rhs (in superposition inferences: neither smaller nor equal).

Superposition Left in Detail:

$$\frac{D' \vee t \approx t' \quad C' \vee s[u] \not\approx s'}{(D' \vee C' \vee s[t']) \not\approx s'}\sigma$$

where  $\sigma = \text{mgu}(t, u)$ ,

$u$  is not a variable,

$t\sigma \not\approx t'\sigma, s\sigma \not\approx s'\sigma$

$(t \approx t')\sigma$  strictly maximal in  $(D' \vee t \approx t')\sigma$ , nothing selected

$(s \not\approx s')\sigma$  maximal in  $(C' \vee s \not\approx s')\sigma$  or selected

Superposition Right in Detail:

$$\frac{D' \vee t \approx t' \quad C' \vee s[u] \approx s'}{(D' \vee C' \vee s[t']) \approx s'}\sigma$$

where  $\sigma = \text{mgu}(t, u)$ ,

$u$  is not a variable,

$t\sigma \not\approx t'\sigma, s\sigma \not\approx s'\sigma$

$(t \approx t')\sigma$  strictly maximal in  $(D' \vee t \approx t')\sigma$ , nothing selected

$(s \approx s')\sigma$  strictly maximal in  $(C' \vee s \approx s')\sigma$ , nothing selected

Equality Resolution in Detail:

$$\frac{C' \vee s \not\approx s'}{C'\sigma}$$

where  $\sigma = \text{mgu}(s, s')$ ,  
 $(s \not\approx s')\sigma$  maximal in  $(C' \vee s \approx s')\sigma$  or selected

Equality Factoring in Detail:

$$\frac{C' \vee s' \approx t' \vee s \approx t}{(C' \vee t \not\approx t' \vee s \approx t)\sigma}$$

where  $\sigma = \text{mgu}(s, s')$ ,  
 $s'\sigma \not\approx t'\sigma, s\sigma \not\approx t\sigma$   
 $(s \approx t)\sigma$  maximal in  $(C' \vee s' \approx t' \vee s \approx t)\sigma$ , nothing selected

A ground clause  $C$  is called *redundant w. r. t. a set of ground clauses  $N$* , if it follows from clauses in  $N$  that are smaller than  $C$ .

A clause is *redundant w. r. t. a set of clauses  $N$* , if all its ground instances are redundant w. r. t.  $G_\Sigma(N)$ .

The set of all clauses that are redundant w. r. t.  $N$  is denoted by  $Red(N)$ .

$N$  is called *saturated up to redundancy*, if the conclusion of every inference from clauses in  $N \setminus Red(N)$  is contained in  $N \cup Red(N)$ .

### Superposition: Refutational Completeness

For a set  $E$  of ground equations,  $T_\Sigma(\emptyset)/E$  is an  $E$ -interpretation (or  $E$ -algebra) with universe  $\{ [t] \mid t \in T_\Sigma(\emptyset) \}$ .

One can show (similar to the proof of Birkhoff's Theorem) that for every *ground* equation  $s \approx t$  we have  $T_\Sigma(\emptyset)/E \models s \approx t$  if and only if  $s \leftrightarrow_E^* t$ .

In particular, if  $E$  is a convergent set of rewrite rules  $R$  and  $s \approx t$  is a ground equation, then  $T_\Sigma(\emptyset)/R \models s \approx t$  if and only if  $s \downarrow_R t$ . By abuse of terminology, we say that an equation or clause is valid (or true) in  $R$  if and only if it is true in  $T_\Sigma(\emptyset)/R$ .

*Construction of candidate interpretations* (Bachmair & Ganzinger 1990):

Let  $N$  be a set of clauses not containing  $\perp$ . Using induction on the clause ordering we define sets of rewrite rules  $E_C$  and  $R_C$  for all  $C \in G_\Sigma(N)$  as follows:

Assume that  $E_D$  has already been defined for all  $D \in G_\Sigma(N)$  with  $D \prec_C C$ . Then  $R_C = \bigcup_{D \prec_C C} E_D$ .

The set  $E_C$  contains the rewrite rule  $s \rightarrow t$ , if

- (a)  $C = C' \vee s \approx t$ .
- (b)  $s \approx t$  is strictly maximal in  $C$ .
- (c)  $s \succ t$ .
- (d)  $C$  is false in  $R_C$ .
- (e)  $C'$  is false in  $R_C \cup \{s \rightarrow t\}$ .
- (f)  $s$  is irreducible w. r. t.  $R_C$ .

In this case,  $C$  is called *productive*. Otherwise  $E_C = \emptyset$ .

Finally,  $R_\infty = \bigcup_{D \in G_\Sigma(N)} E_D$ .

**Lemma 6.5** *If  $E_C = \{s \rightarrow t\}$  and  $E_D = \{u \rightarrow v\}$ , then  $s \succ u$  if and only if  $C \succ_C D$ .*

**Corollary 6.6** *The rewrite systems  $R_C$  and  $R_\infty$  are convergent.*

**Proof.** Obviously,  $s \succ t$  for all rules  $s \rightarrow t$  in  $R_C$  and  $R_\infty$ .

Furthermore, it is easy to check that there are no critical pairs between any two rules: Assume that there are rules  $u \rightarrow v$  in  $E_D$  and  $s \rightarrow t$  in  $E_C$  such that  $u$  is a subterm of  $s$ . As  $\succ$  is a reduction ordering that is total on ground terms, we get  $u \prec s$  and therefore  $D \prec_C C$  and  $E_D \subseteq R_C$ . But then  $s$  would be reducible by  $R_C$ , contradicting condition (f).  $\square$

**Lemma 6.7** *If  $D \preceq_C C$  and  $E_C = \{s \rightarrow t\}$ , then  $s \succ u$  for every term  $u$  occurring in a negative literal in  $D$  and  $s \succeq v$  for every term  $v$  occurring in a positive literal in  $D$ .*

**Corollary 6.8** *If  $D \in G_\Sigma(N)$  is true in  $R_D$ , then  $D$  is true in  $R_\infty$  and  $R_C$  for all  $C \succ_C D$ .*

**Proof.** If a positive literal of  $D$  is true in  $R_D$ , then this is obvious.

Otherwise, some negative literal  $s \not\approx t$  of  $D$  must be true in  $R_D$ , hence  $s \not\downarrow_{R_D} t$ . As the rules in  $R_\infty \setminus R_D$  have left-hand sides that are larger than  $s$  and  $t$ , they cannot be used in a rewrite proof of  $s \downarrow t$ , hence  $s \not\downarrow_{R_C} t$  and  $s \not\downarrow_{R_\infty} t$ .  $\square$

**Corollary 6.9** *If  $D = D' \vee u \approx v$  is productive, then  $D'$  is false and  $D$  is true in  $R_\infty$  and  $R_C$  for all  $C \succ_C D$ .*

**Proof.** Obviously,  $D$  is true in  $R_\infty$  and  $R_C$  for all  $C \succ_C D$ .

Since all negative literals of  $D'$  are false in  $R_D$ , it is clear that they are false in  $R_\infty$  and  $R_C$ . For the positive literals  $u' \approx v'$  of  $D'$ , condition (e) ensures that they are false in  $R_D \cup \{u \rightarrow v\}$ . Since  $u' \preceq u$  and  $v' \preceq u$  and all rules in  $R_\infty \setminus R_D$  have left-hand sides that are larger than  $u$ , these rules cannot be used in a rewrite proof of  $u' \downarrow v'$ , hence  $u' \not\downarrow_{R_C} v'$  and  $u' \not\downarrow_{R_\infty} v'$ .  $\square$

**Lemma 6.10 (“Lifting Lemma”)** *Let  $C$  be a clause and let  $\theta$  be a substitution such that  $C\theta$  is ground. Then every equality resolution or equality factoring inference from  $C\theta$  is a ground instance of an inference from  $C$ .*

**Proof.** Exercise.  $\square$

**Lemma 6.11 (“Lifting Lemma”)** *Let  $D = D' \vee u \approx v$  and  $C = C' \vee [\neg] s \approx t$  be two clauses (without common variables) and let  $\theta$  be a substitution such that  $D\theta$  and  $C\theta$  are ground.*

*If there is a superposition inference between  $D\theta$  and  $C\theta$  where  $u\theta$  and some subterm of  $s\theta$  are overlapped, and  $u\theta$  does not occur in  $s\theta$  at or below a variable position of  $s$ , then the inference is a ground instance of a superposition inference from  $D$  and  $C$ .*

**Proof.** Exercise.  $\square$

**Theorem 6.12 (“Model Construction”)** *Let  $N$  be a set of clauses that is saturated up to redundancy and does not contain the empty clause. Then we have for every ground clause  $C\theta \in G_\Sigma(N)$ :*

- (i)  $E_{C\theta} = \emptyset$  if and only if  $C\theta$  is true in  $R_{C\theta}$ .
- (ii) If  $C\theta$  is redundant w. r. t.  $G_\Sigma(N)$ , then it is true in  $R_{C\theta}$ .
- (iii)  $C\theta$  is true in  $R_\infty$  and in  $R_D$  for every  $D \in G_\Sigma(N)$  with  $D \succ_C C\theta$ .

**Proof.** We use induction on the clause ordering  $\succ_c$  and assume that (i)–(iii) are already satisfied for all clauses in  $G_\Sigma(N)$  that are smaller than  $C\theta$ . Note that the “if” part of (i) is obvious from the construction and that condition (iii) follows immediately from (i) and Corollaries 6.8 and 6.9. So it remains to show (ii) and the “only if” part of (i).