4.4 Termination

Termination problems:

Given a finite TRS R and a term t, are all R-reductions starting from t terminating?

Given a finite TRS R, are all R-reductions terminating?

Proposition 4.12 *Both termination problems for TRSs are undecidable in general.*

Proof. Encode Turing machines using rewrite rules and reduce the (uniform) halting problems for TMs to the termination problems for TRSs. □

Consequence:

Decidable criteria for termination are not complete.

Reduction Orderings

Goal:

Given a finite TRS R, show termination of R by looking at finitely many rules $l \to r \in R$, rather than at infinitely many possible replacement steps $s \to_R s'$.

A binary relation $\sqsupset$ over $\mathrm{T}_\Sigma(X)$ is called *compatible with Σ-operations*, if $s \sqsupset s'$ implies $f(t_1, \dots, s, \dots, t_n) \sqsupset f(t_1, \dots, s', \dots, t_n)$ for all $f \in \Omega$ and $s, s', t_i \in \mathrm{T}_\Sigma(X)$.

Lemma 4.13 *The relation $\sqsupset$ is compatible with Σ-operations, if and only if $s \sqsupset s'$ implies $t[s]_p \sqsupset t[s']_p$ for all $s, s', t \in \mathrm{T}_\Sigma(X)$ and $p \in \mathrm{pos}(t)$.*

Note: *compatible with Σ-operations* = *compatible with contexts*.

A binary relation $\sqsupset$ over $\mathrm{T}_\Sigma(X)$ is called *stable under substitutions*, if $s \sqsupset s'$ implies $s\sigma \sqsupset s'\sigma$ for all $s, s' \in \mathrm{T}_\Sigma(X)$ and substitutions σ.

A binary relation $\sqsupset$ is called a *rewrite relation*, if it is compatible with Σ-operations and stable under substitutions.

Example: If R is a TRS, then $\to_R$ is a rewrite relation.

A strict partial ordering over $\mathrm{T}_\Sigma(X)$ that is a rewrite relation is called *rewrite ordering*.

A well-founded rewrite ordering is called *reduction ordering*.

Theorem 4.14 *A TRS R terminates if and only if there exists a reduction ordering $\succ$ such that $l \succ r$ for every rule $l \to r \in R$.*

Proof. "if": $s \to_R s'$ if and only if $s = t[l\sigma]_p$, $s' = t[r\sigma]_p$. If $l \succ r$, then $l\sigma \succ r\sigma$ and therefore $t[l\sigma]_p \succ t[r\sigma]_p$. This implies $\to_R \subseteq \succ$. Since $\succ$ is a well-founded ordering, $\to_R$ is terminating.

"only if": Define $\succ = \to_R^+$. If $\to_R$ is terminating, then $\succ$ is a reduction ordering. □

Two Different Scenarios

Depending on the application, the TRS whose termination we want to show can be

(i) fixed and known in advance, or

(ii) evolving (e.g., generated by some saturation process).

Methods for case (ii) are also usable for case (i).
Many methods for case (i) are not usable for case (ii).

We will first consider case (ii);
additional techniques for case (i) will be considered later.

The Interpretation Method

Proving termination by interpretation:

Let $\mathcal{A}$ be a Σ-algebra; let $\succ$ be a well-founded strict partial ordering on its universe.

Define the ordering $\succ_{\mathcal{A}}$ over $\mathrm{T}_\Sigma(X)$ by $s \succ_{\mathcal{A}} t$ iff $\mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(t)$ for all assignments $\beta : X \to U_{\mathcal{A}}$.

Is $\succ_{\mathcal{A}}$ a reduction ordering?

Lemma 4.15 *$\succ_{\mathcal{A}}$ is stable under substitutions.*

Proof. Let $s \succ_{\mathcal{A}} s'$, that is, $\mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(s')$ for all assignments $\beta : X \to U_{\mathcal{A}}$. Let σ be a substitution. We have to show that $\mathcal{A}(\gamma)(s\sigma) \succ \mathcal{A}(\gamma)(s'\sigma)$ for all assignments $\gamma : X \to U_{\mathcal{A}}$. Choose $\beta = \gamma \circ \sigma$, then by the substitution lemma, $\mathcal{A}(\gamma)(s\sigma) = \mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(s') = \mathcal{A}(\gamma)(s'\sigma)$. Therefore $s\sigma \succ_{\mathcal{A}} s'\sigma$. □

A function $f : U_{\mathcal{A}}^n \to U_{\mathcal{A}}$ is called *monotone* (with respect to $\succ$), if $a \succ a'$ implies $f(b_1, \dots, a, \dots, b_n) \succ f(b_1, \dots, a', \dots, b_n)$ for all $a, a', b_i \in U_{\mathcal{A}}$.

Lemma 4.16 *If the interpretation $f_{\mathcal{A}}$ of every function symbol f is monotone w. r. t. $\succ$, then $\succ_{\mathcal{A}}$ is compatible with Σ-operations.*

Proof. Let $s \succ s'$, that is, $\mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(s')$ for all $\beta : X \to U_{\mathcal{A}}$. Let $\beta : X \to U_{\mathcal{A}}$ be an arbitrary assignment. Then

$$\begin{aligned}\mathcal{A}(\beta)(f(t_1, \dots, s, \dots, t_n)) &= f_{\mathcal{A}}(\mathcal{A}(\beta)(t_1), \dots, \mathcal{A}(\beta)(s), \dots, \mathcal{A}(\beta)(t_n)) \\ &\succ f_{\mathcal{A}}(\mathcal{A}(\beta)(t_1), \dots, \mathcal{A}(\beta)(s'), \dots, \mathcal{A}(\beta)(t_n)) \\ &= \mathcal{A}(\beta)(f(t_1, \dots, s', \dots, t_n))\end{aligned}$$

Therefore $f(t_1, \dots, s, \dots, t_n) \succ_{\mathcal{A}} f(t_1, \dots, s', \dots, t_n)$. □

Theorem 4.17 *If the interpretation $f_{\mathcal{A}}$ of every function symbol f is monotone w. r. t. $\succ$, then $\succ_{\mathcal{A}}$ is a reduction ordering.*

Proof. By the previous two lemmas, $\succ_{\mathcal{A}}$ is a rewrite relation. If there were an infinite chain $s_1 \succ_{\mathcal{A}} s_2 \succ_{\mathcal{A}} \dots$, then it would correspond to an infinite chain $\mathcal{A}(\beta)(s_1) \succ \mathcal{A}(\beta)(s_2) \succ \dots$ (with β chosen arbitrarily). Thus $\succ_{\mathcal{A}}$ is well-founded. Irreflexivity and transitivity are proved similarly. □

Polynomial Orderings

Polynomial orderings:

Instance of the interpretation method:

The carrier set $U_{\mathcal{A}}$ is $\mathbb{N}$ or some subset of $\mathbb{N}$.

To every function symbol f with arity n we associate a polynomial $P_f(X_1, \dots, X_n) \in \mathbb{N}[X_1, \dots, X_n]$ with coefficients in $\mathbb{N}$ and indeterminates $X_1, \dots, X_n$. Then we define $f_{\mathcal{A}}(a_1, \dots, a_n) = P_f(a_1, \dots, a_n)$ for $a_i \in U_{\mathcal{A}}$.

Requirement 1:

If $a_1, \dots, a_n \in U_{\mathcal{A}}$, then $f_{\mathcal{A}}(a_1, \dots, a_n) \in U_{\mathcal{A}}$. (Otherwise, $\mathcal{A}$ would not be a Σ-algebra.)

Requirement 2:

$f_{\mathcal{A}}$ must be monotone (w. r. t. $\succ$).

From now on:

$U_{\mathcal{A}} = \{ n \in \mathbb{N} \mid n \geq 1 \}$.

If arity$(f) = 0$, then P_f is a constant ≥ 1.

If arity$(f) = n \geq 1$, then P_f is a polynomial $P(X_1, \dots, X_n)$, such that every X_i occurs in some monomial with exponent at least 1 and non-zero coefficient.

$\Rightarrow$ Requirements 1 and 2 are satisfied.

The mapping from function symbols to polynomials can be extended to terms: A term t containing the variables $x_1, \dots, x_n$ yields a polynomial P_t with indeterminates $X_1, \dots, X_n$ (where X_i corresponds to $\beta(x_i)$).

Example:

$\Omega = \{b/0,\ f/1,\ g/3\}$
$P_b = 3, \quad P_f(X_1) = X_1^2, \quad P_g(X_1, X_2, X_3) = X_1 + X_2X_3.$

Let $t = g(f(b), f(x), y)$, then $P_t(X, Y) = 9 + X^2Y$.

If P, Q are polynomials in $\mathbb{N}[X_1, \dots, X_n]$, we write $P > Q$ if $P(a_1, \dots, a_n) > Q(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in U_{\mathcal{A}}$.

Clearly, $l \succ_{\mathcal{A}} r$ iff $P_l > P_r$ iff $P_l - P_r > 0$.

Question: Can we check $P_l - P_r > 0$ automatically?

Hilbert's 10th Problem:

Given a polynomial $P \in \mathbb{Z}[X_1, \dots, X_n]$ with integer coefficients, is $P = 0$ for some n-tuple of natural numbers?

Theorem 4.18 *Hilbert's 10th Problem is undecidable.*

Proposition 4.19 *Given a polynomial interpretation and two terms l, r, it is undecidable whether $P_l > P_r$.*

Proof. By reduction of Hilbert's 10th Problem. □

One easy case:

If we restrict to linear polynomials, deciding whether $P_l - P_r > 0$ is trivial:

$\sum k_i a_i + k > 0$ for all $a_1, \dots, a_n \geq 1$ if and only if

$k_i \geq 0$ for all $i \in \{1, \dots, n\}$,

and $\sum k_i + k > 0$

Another possible solution:

Test whether $P_l(a_1, \dots, a_n) > P_r(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in \{ x \in \mathbb{R} \mid x \geq 1 \}$.

This is decidable (but hard). Since $U_{\mathcal{A}} \subseteq \{ x \in \mathbb{R} \mid x \geq 1 \}$, it implies $P_l > P_r$.

Alternatively:

Use fast overapproximations.

Simplification Orderings

The *proper subterm ordering* $\rhd$ is defined by $s \rhd t$ if and only if $s|_p = t$ for some position $p \neq \varepsilon$ of s.

A rewrite ordering $\succ$ over $\mathrm{T}_\Sigma(X)$ is called *simplification ordering,* if it has the *subterm property:* $s \rhd t$ implies $s \succ t$ for all $s, t \in \mathrm{T}_\Sigma(X)$.

Example:

Let R_{emb} be the rewrite system $R_{\mathrm{emb}} = \{ f(x_1, \dots, x_n) \to x_i \mid f \in \Omega,\, 1 \leq i \leq n = \mathrm{arity}(f) \}$.

Define $\rhd_{\mathrm{emb}} = \to^+_{R_{\mathrm{emb}}}$ and $\unrhd_{\mathrm{emb}} = \to^*_{R_{\mathrm{emb}}}$ *("homeomorphic embedding relation").*

$\rhd_{\mathrm{emb}}$ is a simplification ordering.

Lemma 4.20 *If $\succ$ is a simplification ordering, then $s \rhd_{\mathrm{emb}} t$ implies $s \succ t$ and $s \unrhd_{\mathrm{emb}} t$ implies $s \succeq t$.*

Proof. Since $\succ$ is transitive and $\succeq$ is transitive and reflexive, it suffices to show that $s \to_{R_{\mathrm{emb}}} t$ implies $s \succ t$. By definition, $s \to_{R_{\mathrm{emb}}} t$ if and only if $s = s[l\sigma]$ and $t = s[r\sigma]$ for some rule $l \to r \in R_{\mathrm{emb}}$. Obviously, $l \rhd r$ for all rules in R_{emb}, hence $l \succ r$. Since $\succ$ is a rewrite relation, $s = s[l\sigma] \succ s[r\sigma] = t$. $\square$

Goal:

Show that every simplification ordering is well-founded (and therefore a reduction ordering).

Note: This works only for *finite* signatures!

To fix this for infinite signatures, the definition of simplification orderings and the definition of embedding have to be modified.

Theorem 4.21 ("Kruskal's Theorem") *Let Σ be a finite signature, let X be a finite set of variables. Then for every infinite sequence $t_1, t_2, t_3, \ldots$ there are indices $j > i$ such that $t_j \trianglerighteq_{\text{emb}} t_i$. ($\trianglerighteq_{\text{emb}}$ is called* a well-partial-ordering (wpo).*)*

Proof. See Baader and Nipkow, page 113–115. □

Theorem 4.22 (Dershowitz) *If Σ is a finite signature, then every simplification ordering $\succ$ on $\mathrm{T}_\Sigma(X)$ is well-founded (and therefore a reduction ordering).*

Proof. Suppose that $t_1 \succ t_2 \succ t_3 \succ \ldots$ is an infinite descending chain.

First assume that there is an $x \in \text{vars}(t_{i+1}) \setminus \text{vars}(t_i)$. Let $\sigma = \{x \mapsto t_i\}$, then $t_{i+1}\sigma \trianglerighteq x\sigma = t_i$ and therefore $t_i = t_i\sigma \succ t_{i+1}\sigma \succeq t_i$, contradicting reflexivity.

Consequently, $\text{vars}(t_i) \supseteq \text{vars}(t_{i+1})$ and $t_i \in \mathrm{T}_\Sigma(V)$ for all i, where V is the finite set $\text{vars}(t_1)$. By Kruskal's Theorem, there are $i < j$ with $t_i \trianglelefteq_{\text{emb}} t_j$. Hence $t_i \preceq t_j$, contradicting $t_i \succ t_j$. □

There are reduction orderings that are not simplification orderings and terminating TRSs that are not contained in any simplification ordering.

Example:

Let $R = \{f(f(x)) \to f(g(f(x)))\}$.

R terminates and $\to_R^+$ is therefore a reduction ordering.

Assume that $\to_R$ were contained in a simplification ordering $\succ$. Then $f(f(x)) \to_R f(g(f(x)))$ implies $f(f(x)) \succ f(g(f(x)))$, and $f(g(f(x))) \trianglerighteq_{\text{emb}} f(f(x))$ implies $f(g(f(x))) \succeq f(f(x))$, hence $f(f(x)) \succ f(f(x))$.

Path Orderings

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω.

The *lexicographic path ordering* $\succ_{\text{lpo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{\text{lpo}} t$ iff

(1) $t \in \text{vars}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

  (a) $s_i \succeq_{\text{lpo}} t$ for some i, or

  (b) $f \succ g$ and $s \succ_{\text{lpo}} t_j$ for all j, or

(c) $f = g$, $s \succ_{\text{lpo}} t_j$ for all j, and $(s_1, \ldots, s_m)\ (\succ_{\text{lpo}})_{\text{lex}}\ (t_1, \ldots, t_n)$.

Lemma 4.23 *$s \succ_{\text{lpo}} t$ implies* $\text{vars}(s) \supseteq \text{vars}(t)$.

Proof. By induction on $|s| + |t|$ and case analysis. □

Theorem 4.24 *$\succ_{\text{lpo}}$ is a simplification ordering on $\mathrm{T}_\Sigma(X)$.*

Proof. Show transitivity, subterm property, stability under substitutions, compatibility with Σ-operations, and irreflexivity, usually by induction on the sum of the term sizes and case analysis. Details: Baader and Nipkow, page 119/120. □

Theorem 4.25 *If the precedence $\succ$ is total, then the lexicographic path ordering $\succ_{\text{lpo}}$ is total on ground terms, i. e., for all $s, t \in \mathrm{T}_\Sigma(\emptyset)$: $s \succ_{\text{lpo}} t \vee t \succ_{\text{lpo}} s \vee s = t$.*

Proof. By induction on $|s| + |t|$ and case analysis. □

Recapitulation:

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω. The *lexicographic path ordering* $\succ_{\text{lpo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{\text{lpo}} t$ iff

(1) $t \in \text{vars}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

  (a) $s_i \succeq_{\text{lpo}} t$ for some i, or

  (b) $f \succ g$ and $s \succ_{\text{lpo}} t_j$ for all j, or

  (c) $f = g$, $s \succ_{\text{lpo}} t_j$ for all j, and $(s_1, \ldots, s_m)\ (\succ_{\text{lpo}})_{\text{lex}}\ (t_1, \ldots, t_n)$.

There are several possibilities to compare subterms in (2)(c):

- compare list of subterms lexicographically left-to-right (*"lexicographic path ordering (lpo)"*, Kamin and Lévy)
- compare list of subterms lexicographically right-to-left (or according to some permutation π)
- compare multiset of subterms using the multiset extension (*"multiset path ordering (mpo)"*, Dershowitz)
- to each function symbol f with $\text{arity}(n) \geq 1$ associate a status $\in \{mul\} \cup \{\, lex_\pi \mid \pi : \{1, \ldots, n\} \to \{1, \ldots, n\}\,\}$ and compare according to that status (*"recursive path ordering (rpo) with status"*)

The Knuth-Bendix Ordering

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω, let $w : \Omega \cup X \to \mathbb{R}_0^+$ be a *weight function*, such that the following admissibility conditions are satisfied:

$w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in X$; $w(c) \geq w_0$ for all constants $c \in \Omega$.

If $w(f) = 0$ for some $f \in \Omega$ with $\text{arity}(f) = 1$, then $f \succeq g$ for all $g \in \Omega$.

The weight function w can be extended to terms as follows:

$$w(t) = \sum_{x \in \text{vars}(t)} w(x) \cdot \#(x,t) + \sum_{f \in \Omega} w(f) \cdot \#(f,t).$$

The *Knuth-Bendix ordering* $\succ_{\text{kbo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ and w is defined by: $s \succ_{\text{kbo}} t$ iff

(1) $\#(x,s) \geq \#(x,t)$ for all variables x and $w(s) > w(t)$, or

(2) $\#(x,s) \geq \#(x,t)$ for all variables x, $w(s) = w(t)$, and

- (a) $t = x$, $s = f^n(x)$ for some $n \geq 1$, or
- (b) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f \succ g$, or
- (c) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and $(s_1, \ldots, s_m)\ (\succ_{\text{kbo}})_{\text{lex}}\ (t_1, \ldots, t_m)$.

Theorem 4.26 *The Knuth-Bendix ordering induced by $\succ$ and w is a simplification ordering on $\mathrm{T}_\Sigma(X)$.*

Proof. Baader and Nipkow, pages 125–129. □

Remark

If $\Pi \neq \emptyset$, then all the term orderings described in this section can also be used to compare non-equational atoms by treating predicate symbols like function symbols.

Defining a weight $w(f) = 0$ for some unary function symbol f was in particular introduced for the application of KBO to equational systems defining groups.

4.5 Knuth-Bendix Completion

Completion:

Goal: Given a set E of equations, transform E into an equivalent convergent set R of rewrite rules.
(If R is finite: decision procedure for E.)

How to ensure termination?

Fix a reduction ordering $\succ$ and construct R in such a way that $\rightarrow_R \subseteq \succ$ (i. e., $l \succ r$ for every $l \rightarrow r \in R$).

How to ensure confluence?

Check that all critical pairs are joinable.

Knuth-Bendix Completion: Inference Rules

The completion procedure is itself presented as a set of rewrite rules working on a pair of equations E and rules R: $(E_0; R_0) \Rightarrow (E_1; R_1) \Rightarrow (E_2; R_2) \Rightarrow \ldots$

At the beginning, $E = E_0$ is the input set and $R = R_0$ is empty. At the end, E should be empty; then R is the result.

For each step $(E; R) \Rightarrow (E'; R')$, the equational theories of $E \cup R$ and $E' \cup R'$ agree: $\approx_{E \cup R} = \approx_{E' \cup R'}$.

Notations:

The formula $s \dot{\approx} t$ denotes either $s \approx t$ or $t \approx s$.

$\mathrm{CP}(R)$ denotes the set of all critical pairs between rules in R.

Orient

$$(E \uplus \{s \dot{\approx} t\}; R) \quad \Rightarrow_{KBC} \quad (E; R \cup \{s \rightarrow t\})$$

if $s \succ t$

Note: There are equations $s \approx t$ that cannot be oriented, i. e., neither $s \succ t$ nor $t \succ s$.

Trivial equations cannot be oriented – but we don't need them anyway:

Delete

$$(E \uplus \{s \approx s\}; R) \quad \Rightarrow_{KBC} \quad (E; R)$$

Critical pairs between rules in R are turned into additional equations:

Deduce
$(E; R) \quad \Rightarrow_{KBC} \quad (E \cup \{s \approx t\}; R)$

if $\langle s, t\rangle \in \mathrm{CP}(R)$

Note: If $\langle s, t\rangle \in \mathrm{CP}(R)$ then $s \;{}_R\!\!\leftarrow u \rightarrow_R t$ and hence $R \models s \approx t$.

The following inference rules are not absolutely necessary, but very useful (e. g., to get rid of joinable critical pairs and to deal with equations that cannot be oriented):

Simplify-Eq
$(E \uplus \{s \dot{\approx} t\}; R) \quad \Rightarrow_{KBC} \quad (E \cup \{u \approx t\}; R)$

if $s \rightarrow_R u$

Simplification of the right-hand side of a rule is unproblematic.

R-Simplify-Rule
$(E; R \uplus \{s \rightarrow t\}) \quad \Rightarrow_{KBC} \quad (E; R \cup \{s \rightarrow u\})$

if $t \rightarrow_R u$

Simplification of the left-hand side may influence orientability and orientation. Therefore, it yields an *equation*:

L-Simplify-Rule
$(E; R \uplus \{s \rightarrow t\}) \quad \Rightarrow_{KBC} \quad (E \cup \{u \approx t\}; R$

if $s \rightarrow_R u$ using a rule $l \rightarrow r \in R$ such that $s \sqsupset l$ (see next slide).

For technical reasons, the lhs of $s \rightarrow t$ may only be simplified using a rule $l \rightarrow r$, if $l \rightarrow r$ *cannot* be simplified using $s \rightarrow t$, that is, if $s \sqsupset l$, where the *encompassment quasi-ordering* $\mathrel{\underset{\sim}{\sqsupset}}$ is defined by

$s \mathrel{\underset{\sim}{\sqsupset}} l$ if $s|_p = l\sigma$ for some p and σ

and $\sqsupset = \mathrel{\underset{\sim}{\sqsupset}} \setminus \mathrel{\underset{\sim}{\sqsubset}}$ is the strict part of $\mathrel{\underset{\sim}{\sqsupset}}$.

Lemma 4.27 $\sqsupset$ *is a well-founded strict partial ordering.*

Lemma 4.28 *If* $E, R \vdash E', R'$, *then* $\approx_{E\cup R} = \approx_{E'\cup R'}$.

Lemma 4.29 *If* $E, R \vdash E', R'$ *and* $\rightarrow_R \subseteq \succ$, *then* $\rightarrow_{R'} \subseteq \succ$.

Knuth-Bendix Completion: Correctness Proof

If we run the completion procedure on a set E of equations, different things can happen:

(1) We reach a state where no more inference rules are applicable and E is not empty. $\Rightarrow$ Failure (try again with another ordering?)

(2) We reach a state where E is empty and all critical pairs between the rules in the current R have been checked.

(3) The procedure runs forever.

In order to treat these cases simultaneously, we need some definitions.

A (finite or infinite sequence) $(E_0; R_0) \Rightarrow_{KBC} (E_1; R_1) \Rightarrow_{KBC} (E_2; R_2) \Rightarrow_{KBC} \dots$ with $R_0 = \emptyset$ is called a *run* of the completion procedure with input E_0 and $\succ$.

For a run, $E_\infty = \bigcup_{i \geq 0} E_i$ and $R_\infty = \bigcup_{i \geq 0} R_i$.

The sets of *persistent equations or rules* of the run are $E_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} E_j$ and $R_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} R_j$.

Note: If the run is finite and ends with E_n, R_n, then $E_* = E_n$ and $R_* = R_n$.

A run is called *fair*, if $CP(R_*) \subseteq E_\infty$ (i. e., if every critical pair between persisting rules is computed at some step of the derivation).

Goal:

Show: If a run is fair and E_* is empty, then R_* is convergent and equivalent to E_0.

In particular: If a run is fair and E_* is empty, then $\approx_{E_0} = \approx_{E_\infty \cup R_\infty} = \leftrightarrow^*_{E_\infty \cup R_\infty} = \downarrow_{R_*}$.

General assumptions from now on:

$(E_0; R_0) \Rightarrow_{KBC} (E_1; R_1) \Rightarrow_{KBC} (E_2; R_2) \Rightarrow_{KBC} \dots$
is a fair run.

R_0 and E_* are empty.