3.8 Inference Systems and Proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \ldots, F_n, F_{n+1}),\ n \geq 0,$$

called *inferences*, and written

$$\frac{\overbrace{F_1\ \ldots\ F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}}\ .$$

Clausal inference system: premises and conclusions are clauses. One also considers inference systems over other data structures

Proofs

A *proof* in Γ of a formula F from a a set of formulas N (called *assumptions*) is a sequence $F_1, \ldots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference

$$\frac{F_{i_1}\ \ldots\ F_{i_{n_i}}}{F_i}$$

in Γ, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.

Soundness and Completeness

Provability $\vdash_\Gamma$ of F from N in Γ: $N \vdash_\Gamma F$ $:\Leftrightarrow$ there exists a proof Γ of F from N.

Γ is called *sound* $:\Leftrightarrow$

$$\frac{F_1\ \ldots\ F_n}{F} \in \Gamma \quad \Rightarrow \quad F_1, \ldots, F_n \models F$$

Γ is called *complete* $:\Leftrightarrow$

$$N \models F \quad \Rightarrow \quad N \vdash_\Gamma F$$

Γ is called *refutationally complete* $:\Leftrightarrow$

$$N \models \bot \quad \Rightarrow \quad N \vdash_\Gamma \bot$$

Proposition 3.14

(i) *Let* Γ *be sound. Then* $N \vdash_\Gamma F \Rightarrow N \models F$

(ii) $N \vdash_\Gamma F \Rightarrow$ *there exist finitely many clauses* $F_1, \ldots, F_n \in N$ *such that* $F_1, \ldots, F_n \vdash_\Gamma F$

Proofs as Trees

markings	$\widehat{=}$	formulas			
leaves	$\widehat{=}$	assumptions and axioms			
other nodes	$\widehat{=}$	inferences:	conclusion	$\widehat{=}$	ancestor
			premises	$\widehat{=}$	direct descendants

$$\dfrac{P(f(c)) \qquad \dfrac{\dfrac{\dfrac{P(f(c)) \lor Q(b) \qquad \dfrac{\dfrac{P(f(c)) \lor Q(b) \quad \neg P(f(c)) \lor \neg P(f(c)) \lor Q(b)}{\neg P(f(c)) \lor Q(b) \lor Q(b)}}{\neg P(f(c)) \lor Q(b)}}{Q(b) \lor Q(b)}}{Q(b)} \qquad \neg P(f(c)) \lor \neg Q(b)}{\neg P(f(c))}}{\bot}$$

3.9 Propositional Resolution

We observe that propositional clauses and ground clauses are essentially the same.

In this section we only deal with ground clauses.

The Resolution Calculus *Res*

Resolution inference rule:

$$\frac{D \vee A \qquad \neg A \vee C}{D \vee C}$$

Terminology: $D \vee C$: *resolvent*; A: *resolved atom*

(Positive) factorisation inference rule:

$$\frac{C \vee A \vee A}{C \vee A}$$

These are *schematic inference rules*; for each substitution of the *schematic variables* C, D, and A, by ground clauses and ground atoms, respectively, we obtain an inference.

We treat "$\vee$" as associative and commutative, hence A and $\neg A$ can occur anywhere in the clauses; moreover, when we write $C \vee A$, etc., this includes unit clauses, that is, $C = \bot$.

Sample Refutation

1.	$\neg P(f(c)) \vee \neg P(f(c)) \vee Q(b)$	(given)
2.	$P(f(c)) \vee Q(b)$	(given)
3.	$\neg P(g(b,c)) \vee \neg Q(b)$	(given)
4.	$P(g(b,c))$	(given)
5.	$\neg P(f(c)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(c)) \vee Q(b)$	(Fact. 5.)
7.	$Q(b) \vee Q(b)$	(Res. 2. into 6.)
8.	$Q(b)$	(Fact. 7.)
9.	$\neg P(g(b,c))$	(Res. 8. into 3.)
10.	$\bot$	(Res. 4. into 9.)

Resolution with Implicit Factorization RIF

Factorization can be included in the resolution rule:

$$\frac{D \vee A \vee \ldots \vee A \qquad \neg A \vee C}{D \vee C}$$

Sample refutation for RIF:

1.	$\neg P(f(c)) \vee \neg P(f(c)) \vee Q(b)$	(given)
2.	$P(f(c)) \vee Q(b)$	(given)
3.	$\neg P(g(b,c)) \vee \neg Q(b)$	(given)
4.	$P(g(b,c))$	(given)
5.	$\neg P(f(c)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$Q(b) \vee Q(b) \vee Q(b)$	(Res. 2. into 5.)
7.	$\neg P(g(b,c))$	(Res. 6. into 3.)
8.	$\perp$	(Res. 4. into 7.)

Soundness of Resolution

Theorem 3.15 *Propositional resolution is sound.*

Proof. Let $\mathcal{B} \in \Sigma$-Alg. To be shown:

(i) for resolution: $\mathcal{B} \models D \vee A,\ \mathcal{B} \models C \vee \neg A \ \Rightarrow\ \mathcal{B} \models D \vee C$

(ii) for factorization: $\mathcal{B} \models C \vee A \vee A \ \Rightarrow\ \mathcal{B} \models C \vee A$

(i): Assume premises are valid in $\mathcal{B}$. Two cases need to be considered:
If $\mathcal{B} \models A$, then $\mathcal{B} \models C$, hence $\mathcal{B} \models D \vee C$.
Otherwise, $\mathcal{B} \models \neg A$, then $\mathcal{B} \models D$, and again $\mathcal{B} \models D \vee C$.
(ii): even simpler. □

Note: In propositional logic (ground clauses) we have:

1. $\mathcal{B} \models L_1 \vee \ldots \vee L_n \ \Leftrightarrow\ $ there exists i: $\mathcal{B} \models L_i$.
2. $\mathcal{B} \models A$ or $\mathcal{B} \models \neg A$.

This does not hold for formulas with variables!