1.8 DPLL Iteratively

In practice, there are several changes to the procedure:

The pure literal check is often omitted (it is too expensive).

The branching variable is not chosen randomly.

The algorithm is implemented iteratively;
the backtrack stack is managed explicitly
(it may be possible and useful to backtrack more than one level).

Information is reused by learning.

Branching Heuristics

Choosing the right undefined variable to branch is important for efficiency, but the branching heuristics may be expensive itself.

State of the art: use branching heuristics that need not be recomputed too frequently.

In general: choose variables that occur frequently.

The Deduction Algorithm

For applying the unit rule, we need to know the number of literals in a clause that are not false.

Maintaining this number is expensive, however.

Better approach: *"Two watched literals":*

In each clause, select two (currently undefined) "watched" literals.

For each variable P, keep a list of all clauses in which P is watched and a list of all clauses in which $\neg P$ is watched.

If an undefined variable is set to 0 (or to 1), check all clauses in which P (or $\neg P$) is watched and watch another literal (that is true or undefined) in this clause if possible.

Watched literal information need not be restored upon backtracking.

Conflict Analysis and Learning

Goal: Reuse information that is obtained in one branch in further branches.

Method: *Learning:*

If a conflicting clause is found, derive a new clause from the conflict and add it to the current set of clauses.

Problem: This may produce a large number of new clauses; therefore it may become necessary to delete some of them afterwards to save space.

Backjumping

Related technique:
non-chronological backtracking ("backjumping"):

If a conflict is independent of some earlier branch, try to skip over that backtrack level.

Restart

Runtimes of DPLL-style procedures depend extremely on the choice of branching variables.

If no solution is found within a certain time limit, it can be useful to *restart* from scratch with another choice of branchings (but learned clauses may be kept).

In particular, after learning a unit clause a restart is done.

Formalizing DPLL with Refinements

The DPLL procedure is modelled by a transition relation $\Rightarrow_{\mathrm{DPLL}}$ on a set of states.

States:

- *fail*
- $M \parallel N$,

where M is a *list of annotated literals* and N is a set of clauses.

Annotated literal:

- L: deduced literal, due to unit propagation.
- L^{d}: decision literal (guessed literal).

Unit Propagate:

$M \parallel N \cup \{C \vee L\} \;\Rightarrow_{\text{DPLL}}\; M\, L \parallel N \cup \{C \vee L\}$

if C is false under M and L is undefined under M.

Decide:

$M \parallel N \;\Rightarrow_{\text{DPLL}}\; M\, L^{\text{d}} \parallel N$

if L is undefined under M and contained in N.

Fail:

$M \parallel N \cup \{C\} \;\Rightarrow_{\text{DPLL}}\; \textit{fail}$

if C is false under M and M contains no decision literals.

Backjump:

$M'\, L^{\text{d}}\, M'' \parallel N \;\Rightarrow_{\text{DPLL}}\; M'\, L' \parallel N$

if there is some "backjump clause" $C \vee L'$ such that
$N \models C \vee L'$,
C is false under M', and
L' is undefined under M'.

We will see later that the Backjump rule is always applicable, if the list of literals M contains at least one decision literal and some clause in N is false under M.

There are many possible backjump clauses. One candidate: $\overline{L_1} \vee \ldots \vee \overline{L_n}$, where the L_i are all the decision literals in $M\, L^{\text{d}}\, M'$. (But usually there are better choices.)

Lemma 1.14 *If we reach a state $M \parallel N$ starting from $\emptyset \parallel N$, then:*

(1) M does not contain complementary literals.

(2) Every deduced literal L in M follows from N and decision literals occurring before L in M.

Proof. By induction on the length of the derivation. □

Lemma 1.15 *Every derivation starting from $\emptyset \parallel N$ terminates. (Proof follows)*

Proof. (Idea) Consider a DPLL derivation step $M \parallel N \;\Rightarrow_{\text{DPLL}}\; M' \parallel N'$ and a decomposition $M_0 l_1^d M_1 \ldots l_k^d M_k$ of M (accordingly for M'). Let n be the number of distinct propositional variables in N. Then k, k' and the length of M, M' are always smaller than n. We define $f(M) = n - \text{length}(M)$ and finally

$$M \parallel N \;\succ\; M' \parallel N' \quad \text{if}$$