SU: Main Properties

If $E = x_1 \doteq u_1, \ldots, x_k \doteq u_k$, with x_i pairwise distinct, $x_i \notin var(u_j)$, then E is called an (equational problem in) *solved form* representing the solution $\sigma_E = [u_1/x_1, \ldots, u_k/x_k]$.

Proposition 3.24 *If E is a solved form then σ_E is an mgu of E.*

Theorem 3.25

1. *If $E \Rightarrow_{SU} E'$ then σ is a unifier of E iff σ is a unifier of E'*
2. *If $E \Rightarrow^*_{SU} \bot$ then E is not unifiable.*
3. *If $E \Rightarrow^*_{SU} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.*

Proof. (1) We have to show this for each of the rules. Let's treat the case for the 4th rule here. Suppose σ is a unifier of $x \doteq t$, that is, $x\sigma = t\sigma$. Thus, $\sigma \circ [t/x] = \sigma[x \mapsto t\sigma] = \sigma[x \mapsto x\sigma] = \sigma$. Therefore, for any equation $u \doteq v$ in E: $u\sigma = v\sigma$, iff $u[t/x]\sigma = v[t/x]\sigma$. (2) and (3) follow by induction from (1) using Proposition 3.24. □

Main Unification Theorem

Theorem 3.26 *E is unifiable if and only if there is a most general unifier σ of E, such that σ is idempotent and $dom(\sigma) \cup codom(\sigma) \subseteq var(E)$.*

Problem: *exponential growth* of terms possible

Proof of Theorem 3.26.

- $\Rightarrow_{SU}$ is Noetherian. A suitable lexicographic ordering on the multisets E (with $\bot$ minimal) shows this. Compare in this order:
  1. the number of defined variables (d.h. variables x in equations $x \doteq t$ with $x \notin var(t)$), which also occur outside their definition elsewhere in E;
  2. the multi-set ordering induced by (i) the size (number of symbols) in an equation; (ii) if sizes are equal consider $x \doteq t$ smaller than $t \doteq x$, if $t \notin X$.

□

- A system E that is irreducible w. r. t. $\Rightarrow_{SU}$ is either $\bot$ or a solved form.
- Therefore, reducing any E by SU will end (no matter what reduction strategy we apply) in an irreducible E' having the same unifiers as E, and we can read off the mgu (or non-unifiability) of E from E' (Theorem 3.25, Proposition 3.24).
- σ is idempotent because of the substitution in rule 4. $dom(\sigma) \cup codom(\sigma) \subseteq var(E)$, as no new variables are generated.

Rule Based Polynomial Unification

$$\begin{array}{rll}
t \doteq t, E & \Rightarrow_{PU} & E \\
f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{PU} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) \doteq g(\ldots), E & \Rightarrow_{PU} & \bot \\
x \doteq y, E & \Rightarrow_{PU} & x \doteq y, E[y/x] \\
 & & \text{if } x \in var(E), x \neq y \\
x_1 \doteq t_1, \ldots, x_n \doteq t_n, E & \Rightarrow_{PU} & \bot
\end{array}$$

if there are positions p_i with $t_i/p_i = x_{i+1}, t_n/p_n = x_1$ and some $p_i \neq \epsilon$

$$\begin{array}{rll}
x \doteq t, E & \Rightarrow_{PU} & \bot \\
 & & \text{if } x \neq t, x \in var(t) \\
t \doteq x, E & \Rightarrow_{PU} & x \doteq t, E \\
 & & \text{if } t \notin X \\
x \doteq t, x \doteq s, E & \Rightarrow_{PU} & x \doteq t, t \doteq s, E \\
 & & \text{if } t, s \notin X \text{ and } |t| \leq |s|
\end{array}$$

Properties of PU

Theorem 3.27

1. *If $E \Rightarrow_{PU} E'$ then σ is a unifier of E iff σ is a unifier of E'*
2. *If $E \Rightarrow^*_{PU} \bot$ then E is not unifiable.*
3. *If $E \Rightarrow^*_{PU} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.*

The solved form of $\Rightarrow_{PU}$ is different form the solved form obtained from $\Rightarrow_{SU}$. In order to obtain a unifier, the substitutions generated by the single equations have to be composed.

Lifting Lemma

Lemma 3.28 *Let C and D be variable-disjoint clauses. If*

$$\begin{array}{ccc}
D & & C \\
\downarrow \sigma & & \downarrow \rho \\
\multicolumn{3}{c}{\dfrac{D\sigma \qquad\quad C\rho}{C'}}
\end{array} \qquad \text{[propositional resolution]}$$

then there exists a substitution τ such that

$$\begin{array}{c} \dfrac{D \qquad C}{C''} \\ \big\downarrow \tau \\ C' = C''\tau \end{array} \qquad \textit{[general resolution]}$$

An analogous lifting lemma holds for factorization.

Saturation of Sets of General Clauses

Corollary 3.29 *Let N be a set of general clauses saturated under Res, i. e., $Res(N) \subseteq N$. Then also $G_\Sigma(N)$ is saturated, that is,*

$$Res(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Proof. W.l.o.g. we may assume that clauses in N are pairwise variable-disjoint. (Otherwise make them disjoint, and this renaming process changes neither $Res(N)$ nor $G_\Sigma(N)$.)

Let $C' \in Res(G_\Sigma(N))$, meaning (i) there exist resolvable ground instances $D\sigma$ and $C\rho$ of N with resolvent C', or else (ii) C' is a factor of a ground instance $C\sigma$ of C.

Case (i): By the Lifting Lemma, D and C are resolvable with a resolvent C'' with $C''\tau = C'$, for a suitable substitution τ. As $C'' \in N$ by assumption, we obtain that $C' \in G_\Sigma(N)$.

Case (ii): Similar. □

Herbrand's Theorem

Lemma 3.30 *Let N be a set of Σ-clauses, let $\mathcal{A}$ be an interpretation. Then $\mathcal{A} \models N$ implies $\mathcal{A} \models G_\Sigma(N)$.*

Lemma 3.31 *Let N be a set of Σ-clauses, let $\mathcal{A}$ be a* Herbrand *interpretation. Then $\mathcal{A} \models G_\Sigma(N)$ implies $\mathcal{A} \models N$.*

Theorem 3.32 (Herbrand) *A set N of Σ-clauses is satisfiable if and only if it has a Herbrand model over Σ.*

Proof. The "⇐" part is trivial. For the "⇒" part let $N \not\models \bot$.

$$\begin{aligned} N \not\models \bot &\Rightarrow \bot \notin \mathit{Res}^*(N) \qquad \text{(resolution is sound)} \\ &\Rightarrow \bot \notin G_\Sigma(\mathit{Res}^*(N)) \\ &\Rightarrow I_{G_\Sigma(\mathit{Res}^*(N))} \models G_\Sigma(\mathit{Res}^*(N)) \qquad \text{(Thm. 3.19; Cor. 3.29)} \\ &\Rightarrow I_{G_\Sigma(\mathit{Res}^*(N))} \models \mathit{Res}^*(N) \qquad \text{(Lemma 3.31)} \\ &\Rightarrow I_{G_\Sigma(\mathit{Res}^*(N))} \models N \qquad (N \subseteq \mathit{Res}^*(N)) \quad \Box \end{aligned}$$

The Theorem of Löwenheim-Skolem

Theorem 3.33 (Löwenheim–Skolem) *Let Σ be a countable signature and let S be a set of closed Σ-formulas. Then S is satisfiable iff S has a model over a countable universe.*

Proof. If both X and Σ are countable, then S can be at most countably infinite. Now generate, maintaining satisfiability, a set N of clauses from S. This extends Σ by at most countably many new Skolem functions to Σ'. As Σ' is countable, so is $T_{\Sigma'}$, the universe of Herbrand-interpretations over Σ'. Now apply Theorem 3.32. □

Refutational Completeness of General Resolution

Theorem 3.34 *Let N be a set of general clauses where $\mathit{Res}(N) \subseteq N$. Then*

$$N \models \bot \Leftrightarrow \bot \in N.$$

Proof. Let $\mathit{Res}(N) \subseteq N$. By Corollary 3.29: $\mathit{Res}(G_\Sigma(N)) \subseteq G_\Sigma(N)$

$$\begin{aligned} N \models \bot &\Leftrightarrow G_\Sigma(N) \models \bot \qquad \text{(Lemma 3.30/3.31; Theorem 3.32)} \\ &\Leftrightarrow \bot \in G_\Sigma(N) \qquad \text{(propositional resolution sound and complete)} \\ &\Leftrightarrow \bot \in N \quad \Box \end{aligned}$$

Compactness of Predicate Logic

Theorem 3.35 (Compactness Theorem for First-Order Logic) *Let Φ be a set of first-order formulas. Φ is unsatisfiable $\Leftrightarrow$ some finite subset $\Psi \subseteq \Phi$ is unsatisfiable.*

Proof. The "⇐" part is trivial. For the "⇒" part let Φ be unsatisfiable and let N be the set of clauses obtained by Skolemization and CNF transformation of the formulas in Φ. Clearly $\mathit{Res}^*(N)$ is unsatisfiable. By Theorem 3.34, $\bot \in \mathit{Res}^*(N)$, and therefore $\bot \in \mathit{Res}^n(N)$ for some $n \in \mathbb{N}$. Consequently, $\bot$ has a finite resolution proof B of depth $\leq n$. Choose Ψ as the subset of formulas in Φ such that the corresponding clauses contain the assumptions (leaves) of B. □

3.12 Ordered Resolution with Selection

Motivation: Search space for *Res* *very* large.

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem 3.19) one only needs to resolve and factor maximal atoms
   $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
   $\Rightarrow$ *order restrictions*
2. In the proof, it does not really matter with which negative literal an inference is performed
   $\Rightarrow$ choose a negative literal don't-care-nondeterministically
   $\Rightarrow$ *selection*

Selection Functions

A *selection function* is a mapping

$$S : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \vee \neg A \vee B$$

$$\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$$

Resolution Calculus $Res_S^{\succ}$

In the completeness proof, we talk about (strictly) maximal literals of *ground* clauses.

In the non-ground calculus, we have to consider those literals that correspond to (strictly) maximal literals of ground instances:

Let $\succ$ be a total and well-founded ordering on ground atoms. A literal L is called *[strictly] maximal* in a clause C if and only if there exists a ground substitution σ such that for no other L' in C: $L\sigma \prec L'\sigma$ [$L\sigma \preceq L'\sigma$].

Let $\succ$ be an atom ordering and S a selection function.

$$\frac{D \vee B \qquad C \vee \neg A}{(D \vee C)\sigma} \qquad [\textit{ordered resolution with selection}]$$

if $\sigma = \text{mgu}(A, B)$ and

(i) $B\sigma$ strictly maximal w. r. t. $D\sigma$;

(ii) nothing is selected in D by S;

(iii) either $\neg A$ is selected, or else nothing is selected in $C \vee \neg A$ and $\neg A\sigma$ is maximal in $C\sigma$.

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \qquad [\textit{ordered factoring}]$$

if $\sigma = \text{mgu}(A, B)$ and $A\sigma$ is maximal in $C\sigma$ and nothing is selected in C.

Special Case: Propositional Logic

For ground clauses the resolution inference simplifies to

$$\frac{D \vee A \qquad C \vee \neg A}{D \vee C}$$

if

(i) $A \succ D$;

(ii) nothing is selected in D by. S;

(iii) $\neg A$ is selected in $C \vee \neg A$, or else nothing is selected in $C \vee \neg A$ and $\neg A \succeq \max(C)$.

Note: For positive literals, $A \succ D$ is the same as $A \succ \max(D)$.

Search Spaces Become Smaller

1	$A \vee B$	
2	$A \vee \boxed{\neg B}$	
3	$\neg A \vee B$	
4	$\neg A \vee \boxed{\neg B}$	
5	$B \vee B$	Res 1, 3
6	B	Fact 5
7	$\neg A$	Res 6, 4
8	A	Res 6, 2
9	$\bot$	Res 8, 7

we assume $A \succ B$ and S as indicated by $\boxed{X}$. The maximal literal in a clause is depicted in red.

With this ordering and selection function the refutation proceeds strictly deterministically in this example. Generally, proof search will still be non-deterministic but the search space will be much smaller than with unrestricted resolution.