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May 26, 2010

Tutorials for "Automated Reasoning"
Solution to the exercise sheet 6

Compute the standard and optimized clausal normal forms of the following first-order formulas:

1. $\exists x\, \forall y\, (\forall z\, (P_1(y,\, z) \lor \neg f(x,y) \approx y) \rightarrow (\forall z\, (P_2(y,\, z) \land \neg P_3(x,\, y))))$,
2. $\forall x\, (\forall y\, ((\forall z\, (P(x,w,z)) \rightarrow \exists w\, (Q(x,y,w))) \rightarrow R(x)) \rightarrow S(y))$.

Solution. The required CNFs can be computed by application of the transformations $(\Rightarrow_P^* \Rightarrow_S^* \Rightarrow_K^*)$ and $(\Rightarrow_{NNF}^* \Rightarrow_{MS}^* \Rightarrow_{SK}^* \Rightarrow_P^* \Rightarrow_K^*)$, respectively.

1. $F = \exists x\, \forall y\, (\forall z\, (P_1(y,\, z) \lor \neg f(x,y) \approx y) \rightarrow (\forall z\, (P_2(y,\, z) \land \neg P_3(x,\, y))))$.

Standard CNF:

$$
\begin{array}{ll}
 & \exists x\, \forall y\, (\forall z\, (P_1(y,z) \lor \neg f(x,y) \approx y) \rightarrow (\forall z\, (P_2(y,z) \land \neg P_3(x,y)))) \\
\Rightarrow_P & \exists x\, \forall y\, \exists z_1\, ((P_1(y,z_1) \lor \neg f(x,y) \approx y) \rightarrow (\forall z\, (P_2(y,z) \land \neg P_3(x,y)))) \\
\Rightarrow_P & \exists x\, \forall y\, \exists z_1\, \forall z_2\, ((P_1(y,z_1) \lor \neg f(x,y) \approx y) \rightarrow (P_2(y,z_2) \land \neg P_3(x,y))) \\
\Rightarrow_S & \forall y\, \exists z_1\, \forall z_2\, ((P_1(y,z_1) \lor \neg f(skf_x,y) \approx y) \rightarrow (P_2(y,z_2) \land \neg P_3(skf_x,y))) \\
\Rightarrow_S & \forall y\, \forall z_2\, ((P_1(y,skf_{z_1}(y)) \lor \neg f(skf_x,y) \approx y) \rightarrow (P_2(y,z_2) \land \neg P_3(skf_x,y))) \\
\Rightarrow_K & \forall y\, \forall z_2\, (\neg(P_1(y,skf_{z_1}(y)) \lor \neg f(skf_x,y) \approx y) \lor (P_2(y,z_2) \land \neg P_3(skf_x,y))) \\
\Rightarrow_K & \forall y\, \forall z_2\, ((\neg P_1(y,skf_{z_1}(y)) \land \neg\neg f(skf_x,y) \approx y) \lor (P_2(y,z_2) \land \neg P_3(skf_x,y))) \\
\Rightarrow_K & \forall y\, \forall z_2\, ((\neg P_1(y,skf_{z_1}(y)) \land f(skf_x,y) \approx y) \lor (P_2(y,z_2) \land \neg P_3(skf_x,y))) \\
\Rightarrow_K^+ & \forall y\, \forall z_2\, ((\neg P_1(y,skf_{z_1}(y)) \lor P_2(y,z_2)) \land (\neg P_1(y,skf_{z_1}(y)) \lor \neg P_3(skf_x,y)) \land \\
 & \quad \land (f(skf_x,y) \approx y \lor P_2(y,z_2)) \land (f(skf_x,y) \approx y \lor \neg P_3(skf_x,y))),
\end{array}
$$

where $skf_x(y_1,\ldots,y_n)$ stands to denote the skolem function (constant, if $n=0$) resulting from skolemization of the variable x.

Hence, the CNF of the input formula F is

$$
\begin{array}{lcl}
\mathrm{CNF}(F) & = & \{\neg P_1(y,skf_{z_1}(y)) \lor P_2(y,z_2), \\
 & & \neg P_1(y,skf_{z_1}(y)) \lor \neg P_3(skf_x,y), \\
 & & f(skf_x,y) \approx y \lor P_2(y,z_2), \\
 & & f(skf_x,y) \approx y \lor \neg P_3(skf_x,y)\}.
\end{array}
$$

Optimized CNF:

$$
\begin{array}{ll}
 & \exists x\, \forall y\, (\forall z\, (P_1(y,z) \vee \neg f(x,y) \approx y) \rightarrow (\forall z\, (P_2(y,z) \wedge \neg P_3(x,y)))) \\
\Rightarrow_{NNF} & \exists x\, \forall y\, (\neg \forall z\, (P_1(y,z) \vee \neg f(x,y) \approx y) \vee \forall z\, (P_2(y,z) \wedge \neg P_3(x,y))) \\
\Rightarrow_{NNF} & \exists x\, \forall y\, (\exists z\, \neg (P_1(y,z) \vee \neg f(x,y) \approx y) \vee \forall z\, (P_2(y,z) \wedge \neg P_3(x,y))) \\
\Rightarrow^{+}_{NNF} & \exists x\, \forall y\, (\exists z\, (\neg P_1(y,z) \wedge f(x,y) \approx y) \vee \forall z\, (P_2(y,z) \wedge \neg P_3(x,y))) \\
\Rightarrow_{MS} & \exists x\, \forall y\, (\exists z\, (\neg P_1(y,z) \wedge f(x,y) \approx y) \vee (\forall z\, P_2(y,z) \wedge \neg P_3(x,y))) \\
\Rightarrow_{MS} & \exists x\, \forall y\, ((\exists z\, \neg P_1(y,z) \wedge f(x,y) \approx y) \vee (\forall z\, P_2(y,z) \wedge \neg P_3(x,y))) \\
 & \quad \text{rename variables:} \\
 & \exists x_1\, \forall y_1\, ((\exists z_1\, \neg P_1(y_1,z_1) \wedge f(x_1,y_1) \approx y_1) \vee (\forall z_2\, P_2(y_1,z_2) \wedge \neg P_3(x_1,y_1))) \\
\Rightarrow_{SK} & \forall y_1\, ((\exists z_1\, \neg P_1(y_1,z_1) \wedge f(\mathit{skf}_{x_1},y_1) \approx y_1) \vee (\forall z_2\, P_2(y_1,z_2) \wedge \neg P_3(\mathit{skf}_{x_1},y_1))) \\
\Rightarrow_{SK} & \forall y_1\, ((\neg P_1(y_1,\mathit{skf}_{z_1}(y_1)) \wedge f(\mathit{skf}_{x_1},y_1) \approx y_1) \vee (\forall z_2\, P_2(y_1,z_2) \wedge \neg P_3(\mathit{skf}_{x_1},y_1))) \\
\Rightarrow_{P} & \forall y_1\, ((\neg P_1(y_1,\mathit{skf}_{z_1}(y_1)) \wedge f(\mathit{skf}_{x_1},y_1) \approx y_1) \vee \forall z_3\, (P_2(y_1,z_3) \wedge \neg P_3(\mathit{skf}_{x_1},y_1))) \\
\Rightarrow_{P} & \forall y_1\, \forall z'\, ((\neg P_1(y_1,\mathit{skf}_{z_1}(y_1)) \wedge f(\mathit{skf}_{x_1},y_1) \approx y_1) \vee (P_2(y_1,z') \wedge \neg P_3(\mathit{skf}_{x_1},y_1))) \\
\Rightarrow^{+}_{K} & \forall y_1\, \forall z'\, ((\neg P_1(y_1,\mathit{skf}_{z_1}(y_1)) \vee P_2(y_1,z')) \wedge (\neg P_1(y_1,\mathit{skf}_{z_1}(y_1)) \vee \neg P_3(\mathit{skf}_{x_1},y_1)) \wedge \\
 & \quad \wedge (f(\mathit{skf}_{x_1},y_1) \approx y_1 \vee P_2(y_1,z')) \wedge (f(\mathit{skf}_{x_1},y_1) \approx y_1 \vee \neg P_3(\mathit{skf}_{x_1},y_1))).
\end{array}
$$

Hence, the CNF of the input formula F is

$$
\begin{array}{rcl}
\mathrm{CNF}(F) & = & \{\neg P_1(y_1,\mathit{skf}_{z_1}(y_1)) \vee P_2(y_1,z'), \\
 & & \;\neg P_1(y_1,\mathit{skf}_{z_1}(y_1)) \vee \neg P_3(\mathit{skf}_{x_1},y_1), \\
 & & \;f(\mathit{skf}_{x_1},y_1) \approx y_1 \vee P_2(y_1,z'), \\
 & & \;f(\mathit{skf}_{x_1},y_1) \approx y_1 \vee \neg P_3(\mathit{skf}_{x_1},y_1)\}.
\end{array}
$$

2. $F = \forall x\, (\forall y\, ((\forall z\, (P(x,w,z)) \rightarrow \exists w\, (Q(x,y,w))) \rightarrow R(x)) \rightarrow S(y))$.

Standard CNF.

$$
\begin{array}{ll}
 & \forall x\, (\forall y\, ((\forall z\, (P(x,w,z)) \rightarrow \exists w\, (Q(x,y,w))) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow_{P} & \forall x\, (\forall y\, (\exists z_1\, (P(x,w,z_1) \rightarrow \exists w\, (Q(x,y,w))) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow_{P} & \forall x\, (\forall y\, (\exists z_1\, \exists w_1\, (P(x,w,z_1) \rightarrow Q(x,y,w_1)) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow_{P} & \forall x\, (\forall y\, \forall z_2\, (\exists w_1\, (P(x,w,z_2) \rightarrow Q(x,y,w_1)) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow_{P} & \forall x\, (\forall y\, \forall z_2\, \forall w_2\, ((P(x,w,z_2) \rightarrow Q(x,y,w_2)) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow_{P} & \forall x\, \exists y'\, (\forall z_2\, \forall w_2\, ((P(x,w,z_2) \rightarrow Q(x,y',w_2)) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow^{+}_{P} & \forall x\, \exists y'\, \exists z'\, \exists w'\, (((P(x,w,z') \rightarrow Q(x,y',w')) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow_{S} & \forall x\, \exists z'\, \exists w'\, (((P(x,w,z') \rightarrow Q(x,\mathit{skf}_{y'}(x,y,w),w')) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow^{+}_{S} & \forall x\, (((P(x,w,\mathit{skf}_{z'}(x,y,w)) \rightarrow Q(x,\mathit{skf}_{y'}(x,y,w),\mathit{skf}_{w'}(x,y,w))) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow^{+}_{S} & \forall x\, (((P(x,w,\mathit{skf}_{z'}(x,y,w)) \rightarrow Q(x,\mathit{skf}_{y'}(x,y,w),\mathit{skf}_{w'}(x,y,w))) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow_{K} & \forall x\, (((\neg P(x,w,\mathit{skf}_{z'}(x,y,w)) \vee Q(x,\mathit{skf}_{y'}(x,y,w),\mathit{skf}_{w'}(x,y,w))) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow^{+}_{K} & \forall x\, (\neg(\neg(\neg P(x,w,\mathit{skf}_{z'}(x,y,w)) \vee Q(x,\mathit{skf}_{y'}(x,y,w),\mathit{skf}_{w'}(x,y,w))) \vee R(x)) \vee S(y)) \\
\Rightarrow_{K} & \forall x\, (\neg((\neg\neg P(x,w,\mathit{skf}_{z'}(x,y,w)) \wedge \neg Q(x,\mathit{skf}_{y'}(x,y,w),\mathit{skf}_{w'}(x,y,w))) \vee R(x)) \vee S(y)) \\
\Rightarrow_{K} & \forall x\, (\neg(\neg\neg P(x,w,\mathit{skf}_{z'}(x,y,w)) \wedge \neg Q(x,\mathit{skf}_{y'}(x,y,w),\mathit{skf}_{w'}(x,y,w))) \wedge \neg R(x)) \vee S(y)) \\
\Rightarrow_{K} & \forall x\, (\neg\neg\neg P(x,w,\mathit{skf}_{z'}(x,y,w)) \vee \neg\neg Q(x,\mathit{skf}_{y'}(x,y,w),\mathit{skf}_{w'}(x,y,w))) \wedge \neg R(x)) \vee S(y)) \\
\Rightarrow_{K} & \forall x\, (\neg P(x,w,\mathit{skf}_{z'}(x,y,w)) \vee Q(x,\mathit{skf}_{y'}(x,y,w),\mathit{skf}_{w'}(x,y,w))) \wedge \neg R(x)) \vee S(y)) \\
\Rightarrow_{K} & \forall x\, ((\neg P(x,w,\mathit{skf}_{z'}(x,y,w)) \vee Q(x,\mathit{skf}_{y'}(x,y,w),\mathit{skf}_{w'}(x,y,w)) \vee S(y)) \wedge (\neg R(x) \vee S(y))).
\end{array}
$$

Hence, the CNF of the input formula F is

$$
\begin{array}{rcl}
\mathrm{CNF}(F) & = & \{\neg P(x,w,\mathit{skf}_{z'}(x,y,w)) \vee Q(x,\mathit{skf}_{y'}(x,y,w),\mathit{skf}_{w'}(x,y,w)) \vee S(y), \\
 & & \;\neg R(x) \vee S(y)\}.
\end{array}
$$

Optimized CNF.

$$
\begin{array}{ll}
& \forall x\,(\forall y\,((\forall z\,P(x,w,z) \rightarrow \exists w\,Q(x,y,w)) \rightarrow R(x)) \rightarrow S(y)) \\
\Rightarrow_{NNF} & \forall x\,(\neg(\forall y\,((\forall z\,P(x,w,z) \rightarrow \exists w\,Q(x,y,w)) \rightarrow R(x))) \vee S(y)) \\
\Rightarrow_{NNF} & \forall x\,(\neg(\forall y\,(\neg(\forall z\,P(x,w,z) \rightarrow \exists w\,Q(x,y,w)) \vee R(x))) \vee S(y)) \\
\Rightarrow_{NNF} & \forall x\,(\neg(\forall y\,(\neg(\neg(\forall z\,P(x,w,z)) \vee \exists w\,Q(x,y,w)) \vee R(x))) \vee S(y)) \\
\Rightarrow_{NNF} & \forall x\,(\exists y\,\neg(\neg(\neg(\forall z\,P(x,w,z)) \vee \exists w\,Q(x,y,w)) \vee R(x)) \vee S(y)) \\
\Rightarrow_{NNF} & \forall x\,(\exists y\,(\neg\neg(\neg(\forall z\,P(x,w,z)) \vee \exists w\,Q(x,y,w)) \wedge \neg R(x)) \vee S(y)) \\
\Rightarrow_{NNF} & \forall x\,(\exists y\,((\neg(\forall z\,P(x,w,z)) \vee \exists w\,Q(x,y,w)) \wedge \neg R(x)) \vee S(y)) \\
\Rightarrow_{NNF} & \forall x\,(\exists y\,((\exists z\,\neg P(x,w,z) \vee \exists w\,Q(x,y,w)) \wedge \neg R(x)) \vee S(y)) \\
\Rightarrow_{MS} & \forall x\,((\exists y\,(\exists z\,\neg P(x,w,z) \vee \exists w\,Q(x,y,w)) \wedge \neg R(x)) \vee S(y)) \\
\Rightarrow_{MS} & \forall x\,(((\exists z\,\neg P(x,w,z) \vee \exists y\,\exists w\,Q(x,y,w)) \wedge \neg R(x)) \vee S(y)) \\
\Rightarrow_{MS} & \forall x\,((\exists z\,\neg P(x,w,z) \vee \exists y\,\exists w\,Q(x,y,w)) \wedge \neg R(x)) \vee S(y) \\
\Rightarrow_{MS} & (\forall x\,(\exists z\,\neg P(x,w,z) \vee \exists y\,\exists w\,Q(x,y,w)) \wedge \forall x\,\neg R(x)) \vee S(y) \\
& \quad \text{rename variables:} \\
\Rightarrow_{MS} & (\forall x_1\,(\exists z_1\,\neg P(x_1,w,z_1) \vee \exists y_1\,\exists w_1\,Q(x_1,y_1,w_1)) \wedge \forall x_2\,\neg R(x_2)) \vee S(y) \\
\Rightarrow_{SK} & (\forall x_1\,(\neg P(x_1,w,skf_{z_1}(x_1,y,w)) \vee \exists y_1\,\exists w_1\,Q(x_1,y_1,w_1)) \wedge \forall x_2\,\neg R(x_2)) \vee S(y) \\
\Rightarrow_{SK} & (\forall x_1\,(\neg P(x_1,w,skf_{z_1}(x_1,y,w)) \vee \exists w_1\,Q(x_1,skf_{y_1}(x_1,y,w),w_1)) \wedge \forall x_2\,\neg R(x_2)) \vee S(y) \\
\Rightarrow_{SK} & (\forall x_1\,(\neg P(x_1,w,skf_{z_1}(x_1,y,w)) \vee Q(x_1,skf_{y_1}(x_1,y,w),skf_{w_1}(x_1,y,w))) \wedge \forall x_2\,\neg R(x_2)) \vee S(y) \\
\Rightarrow_P^+ & \forall x_1'\,(((\neg P(x_1',w,skf_{z_1}(x_1',y,w)) \vee Q(x_1',skf_{y_1}(x_1',y,w),skf_{w_1}(x_1',y,w))) \wedge \forall x_2\,\neg R(x_2)) \vee S(y)) \\
\Rightarrow_P^+ & \forall x_1'\,\forall x_2'\,(((\neg P(x_1',w,skf_{z_1}(x_1',y,w)) \vee Q(x_1',skf_{y_1}(x_1',y,w),skf_{w_1}(x_1',y,w))) \wedge \neg R(x_2')) \vee S(y)) \\
\Rightarrow_K & \forall x_1'\,\forall x_2'\,((\neg P(x_1',w,skf_{z_1}(x_1',y,w)) \vee Q(x_1',skf_{y_1}(x_1',y,w),skf_{w_1}(x_1',y,w)) \vee S(y)) \wedge \\
& \wedge (\neg R(x_2') \vee S(y)))
\end{array}
$$

Hence, the CNF of the input formula F is

$$
\begin{array}{rcl}
\mathrm{CNF}(F) & = & \{\neg P(x_1',w,skf_{z_1}(x_1',y,w)) \vee Q(x_1',skf_{y_1}(x_1',y,w),skf_{w_1}(x_1',y,w)) \vee S(y), \\
& & \neg R(x_2') \vee S(y)\}.
\end{array}
$$