Corollary 4.49 *If $D = D' \vee u \approx v$ is productive, then D' is false and D is true in R_∞ and R_C for all $C \succ_C D$.*

Proof. Obviously, D is true in R_∞ and R_C for all $C \succ_C D$.

Since all negative literals of D' are false in R_D, it is clear that they are false in R_∞ and R_C. For the positive literals $u' \approx v'$ of D', condition (e) ensures that they are false in $R_D \cup \{u \to v\}$. Since $u' \preceq u$ and $v' \preceq u$ and all rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than u, these rules cannot be used in a rewrite proof of $u' \downarrow v'$, hence $u' \not\downarrow_{R_C} v'$ and $u' \not\downarrow_{R_\infty} v'$. □

Lemma 4.50 ("Lifting Lemma") *Let C be a clause and let θ be a substitution such that $C\theta$ is ground. Then every equality resolution or equality factoring inference from $C\theta$ is a ground instance of an inference from C.*

Proof. Exercise. □

Lemma 4.51 ("Lifting Lemma") *Let $D = D' \vee u \approx v$ and $C = C' \vee [\neg]\, s \approx t$ be two clauses (without common variables) and let θ be a substitution such that $D\theta$ and $C\theta$ are ground.*

If there is a superposition inference between $D\theta$ and $C\theta$ where $u\theta$ and some subterm of $s\theta$ are overlapped, and $u\theta$ does not occur in $s\theta$ at or below a variable position of s, then the inference is a ground instance of a superposition inference from D and C.

Proof. Exercise. □

Theorem 4.52 ("Model Construction") *Let N be a set of clauses that is saturated up to redundancy and does not contain the empty clause. Then we have for every ground clause $C\theta \in G_\Sigma(N)$:*

(i) $E_{C\theta} = \emptyset$ if and only if $C\theta$ is true in $R_{C\theta}$.

(ii) If $C\theta$ is redundant w. r. t. $G_\Sigma(N)$, then it is true in $R_{C\theta}$.

(iii) $C\theta$ is true in R_∞ and in R_D for every $D \in G_\Sigma(N)$ with $D \succ_C C\theta$.

Proof. We use induction on the clause ordering $\succ_{\mathrm{C}}$ and assume that (i)–(iii) are already satisfied for all clauses in $G_\Sigma(N)$ that are smaller than $C\theta$. Note that the "if" part of (i) is obvious from the construction and that condition (iii) follows immediately from (i) and Corollaries 4.48 and 4.49. So it remains to show (ii) and the "only if" part of (i).

Case 1: $C\theta$ is redundant w. r. t. $G_\Sigma(N)$.

If $C\theta$ is redundant w. r. t. $G_\Sigma(N)$, then it follows from clauses in $G_\Sigma(N)$ that are smaller than $C\theta$. By part (iii) of the induction hypothesis, these clauses are true in $R_{C\theta}$. Hence $C\theta$ is true in $R_{C\theta}$.

Case 2: $x\theta$ is reducible by $R_{C\theta}$.

Suppose there is a variable x occurring in C such that $x\theta$ is reducible by $R_{C\theta}$, say $x\theta \to_{R_{C\theta}} w$. Let the substitution θ' be defined by $x\theta' = w$ and $y\theta' = y\theta$ for every variable $y \neq x$. The clause $C\theta'$ is smaller than $C\theta$. By part (iii) of the induction hypothesis, it is true in $R_{C\theta}$. By congruence, every literal of $C\theta$ is true in $R_{C\theta}$ if and only if the corresponding literal of $C\theta'$ is true in $R_{C\theta}$; hence $C\theta$ is true in $R_{C\theta}$.

Case 3: $C\theta$ contains a maximal negative literal.

Suppose that $C\theta$ does not fall into Case 1 or 2 and that $C\theta = C'\theta \ \vee\ s\theta \not\approx s'\theta$, where $s\theta \not\approx s'\theta$ is maximal in $C\theta$. If $s\theta \approx s'\theta$ is false in $R_{C\theta}$, then $C\theta$ is clearly true in $R_{C\theta}$ and we are done. So assume that $s\theta \approx s'\theta$ is true in $R_{C\theta}$, that is, $s\theta \downarrow_{R_{C\theta}} s'\theta$. Without loss of generality, $s\theta \succeq s'\theta$.

Case 3.1: $s\theta = s'\theta$.

If $s\theta = s'\theta$, then there is an *equality resolution* inference

$$\frac{C'\theta \ \vee\ s\theta \not\approx s'\theta}{C'\theta}\ .$$

As shown in the Lifting Lemma, this is an instance of an *equality resolution* inference

$$\frac{C' \ \vee\ s \not\approx s'}{C'\sigma}$$

where $C = C' \ \vee\ s \not\approx s'$ is contained in N and $\theta = \rho \circ \sigma$. (Without loss of generality, σ is idempotent, therefore $C'\theta = C'\sigma\rho = C'\sigma\sigma\rho = C'\sigma\theta$, so $C'\theta$ is a ground instance of $C'\sigma$.) Since $C\theta$ is not redundant w. r. t. $G_\Sigma(N)$, C is not redundant w. r. t. N. As N is saturated up to redundancy, the conclusion $C'\sigma$ of the inference from C is contained in $N \cup \mathit{Red}(N)$. Therefore, $C'\theta$ is either contained in $G_\Sigma(N)$ and smaller than $C\theta$, or it follows from clauses in $G_\Sigma(N)$ that are smaller than itself (and therefore smaller than $C\theta$). By the induction hypothesis, clauses in $G_\Sigma(N)$ that are smaller than $C\theta$ are true in $R_{C\theta}$, thus $C'\theta$ and $C\theta$ are true in $R_{C\theta}$.