Theorem 4.35 $\succ_{\rm lpo}$ *is a simplification ordering on* $\mathrm{T}_\Sigma(X)$.

Proof. Show transitivity, subterm property, stability under substitutions, compatibility with Σ-operations, and irreflexivity, usually by induction on the sum of the term sizes and case analysis. Details: Baader and Nipkow, page 119/120. $\square$

Theorem 4.36 *If the precedence* $\succ$ *is total, then the lexicographic path ordering* $\succ_{\rm lpo}$ *is total on ground terms, i.e., for all* $s, t \in \mathrm{T}_\Sigma(\emptyset)$: $s \succ_{\rm lpo} t \vee t \succ_{\rm lpo} s \vee s = t$.

Proof. By induction on $|s| + |t|$ and case analysis. $\square$

Recapitulation:

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω. The *lexicographic path ordering* $\succ_{\rm lpo}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{\rm lpo} t$ iff

(1) $t \in \mathrm{var}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and

- (a) $s_i \succeq_{\rm lpo} t$ for some i, or
- (b) $f \succ g$ and $s \succ_{\rm lpo} t_j$ for all j, or
- (c) $f = g$, $s \succ_{\rm lpo} t_j$ for all j, and $(s_1, \dots, s_m)\ (\succ_{\rm lpo})_{\rm lex}\ (t_1, \dots, t_n)$.

There are several possibilities to compare subterms in (2)(c):

compare list of subterms lexicographically left-to-right (*"lexicographic path ordering (lpo)"*, Kamin and Lévy)

compare list of subterms lexicographically right-to-left (or according to some permutation π)

compare multiset of subterms using the multiset extension (*"multiset path ordering (mpo)"*, Dershowitz)

to each function symbol f with $\mathsf{arity}(n) \geq 1$ associate a status $\in \{mul\} \cup \{\, lex_\pi \mid \pi : \{1, \dots, n\} \to \{1, \dots, n\}\,\}$ and compare according to that status (*"recursive path ordering (rpo) with status"*)

The Knuth-Bendix Ordering

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering *("precedence")* on Ω, let $w : \Omega \cup X \to \mathbb{R}_0^+$ be a *weight function*, such that the following admissibility conditions are satisfied:

$w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in X$; $w(c) \geq w_0$ for all constants $c \in \Omega$.

If $w(f) = 0$ for some $f \in \Omega$ with $\mathsf{arity}(f) = 1$, then $f \succeq g$ for all $g \in \Omega$.

The weight function w can be extended to terms as follows:

$$w(t) = \sum_{x \in \mathrm{var}(t)} w(x) \cdot \#(x,t) + \sum_{f \in \Omega} w(f) \cdot \#(f,t).$$

The *Knuth-Bendix ordering* $\succ_{\mathrm{kbo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ and w is defined by: $s \succ_{\mathrm{kbo}} t$ iff

(1) $\#(x,s) \geq \#(x,t)$ for all variables x and $w(s) > w(t)$, or

(2) $\#(x,s) \geq \#(x,t)$ for all variables x, $w(s) = w(t)$, and

- (a) $t = x$, $s = f^n(x)$ for some $n \geq 1$, or
- (b) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f \succ g$, or
- (c) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and $(s_1, \ldots, s_m)\ (\succ_{\mathrm{kbo}})_{\mathrm{lex}}\ (t_1, \ldots, t_m)$.

Theorem 4.37 *The Knuth-Bendix ordering induced by $\succ$ and w is a simplification ordering on $\mathrm{T}_\Sigma(X)$.*

Proof. Baader and Nipkow, pages 125–129. □