Let E be a set of equations over $\mathrm{T}_\Sigma(X)$. The following inference system allows to derive consequences of E:

$E \vdash t \approx t$ *(Reflexivity)*

$\dfrac{E \vdash t \approx t'}{E \vdash t' \approx t}$ *(Symmetry)*

$\dfrac{E \vdash t \approx t' \qquad E \vdash t' \approx t''}{E \vdash t \approx t''}$ *(Transitivity)*

$\dfrac{E \vdash t_1 \approx t'_1 \quad \dots \quad E \vdash t_n \approx t'_n}{E \vdash f(t_1, \dots, t_n) \approx f(t'_1, \dots, t'_n)}$ *(Congruence)*

$E \vdash t\sigma \approx t'\sigma$ *(Instance)*
  if $(t \approx t') \in E$ and $\sigma : X \to \mathrm{T}_\Sigma(X)$

Lemma 4.11 *The following properties are equivalent:*

(i) $s \leftrightarrow_E^* t$

(ii) $E \vdash s \approx t$ *is derivable.*

(Proof Scetch Follows)

Proof. (i)⇒(ii): $s \leftrightarrow_E t$ implies $E \vdash s \approx t$ by induction on the depth of the position where the rewrite rule is applied; then $s \leftrightarrow_E^* t$ implies $E \vdash s \approx t$ by induction on the number of rewrite steps in $s \leftrightarrow_E^* t$.

(ii)⇒(i): By induction on the size (number of symbols) of the derivation for $E \vdash s \approx t$. □

Constructing a *quotient algebra:*

Let X be a set of variables.

For $t \in \mathrm{T}_\Sigma(X)$ let $[t] = \{\, t' \in \mathrm{T}_\Sigma(X) \mid E \vdash t \approx t' \,\}$ be the *congruence class* of t.

Define a Σ-algebra $\mathrm{T}_\Sigma(X)/E$ (abbreviated by $\mathcal{T}$) as follows:

$U_\mathcal{T} = \{\, [t] \mid t \in \mathrm{T}_\Sigma(X) \,\}$.

$f_\mathcal{T}([t_1], \dots, [t_n]) = [f(t_1, \dots, t_n)]$ for $f \in \Omega$.

Lemma 4.12 $f_\mathcal{T}$ *is well-defined: If* $[t_i] = [t'_i]$, *then* $[f(t_1, \dots, t_n)] = [f(t'_1, \dots, t'_n)]$.

Proof. Follows directly from the *Congruence* rule for $\vdash$. □

Lemma 4.13 $\mathcal{T} = \mathrm{T}_\Sigma(X)/E$ *is an* E*-algebra. (Proof Follows)*

Proof. Let $\forall x_1 \ldots x_n(s \approx t)$ be an equation in E; let β be an arbitrary assignment.

We have to show that $\mathcal{T}(\beta)(\forall\vec{x}(s \approx t)) = 1$, or equivalently, that $\mathcal{T}(\gamma)(s) = \mathcal{T}(\gamma)(t)$ for all $\gamma = \beta[\, x_i \mapsto [t_i] \mid 1 \leq i \leq n\,]$ with $[t_i] \in U_{\mathcal{T}}$.

Let $\sigma = [t_1/x_1, \ldots, t_n/x_n]$, then $s\sigma \in \mathcal{T}(\gamma)(s)$ and $t\sigma \in \mathcal{T}(\gamma)(t)$.

By the *Instance* rule, $E \vdash s\sigma \approx t\sigma$ is derivable, hence $\mathcal{T}(\gamma)(s) = [s\sigma] = [t\sigma] = \mathcal{T}(\gamma)(t)$. □

Lemma 4.14 *Let* X *be a countably infinite set of variables; let* $s, t \in \mathrm{T}_\Sigma(X)$*. If* $\mathrm{T}_\Sigma(X)/E \models \forall\vec{x}(s \approx t)$*, then* $E \vdash s \approx t$ *is derivable. (Proof Follows)*

Proof. Assume that $\mathcal{T} \models \forall\vec{x}(s \approx t)$, i. e., $\mathcal{T}(\beta)(\forall\vec{x}(s \approx t)) = 1$. Consequently, $\mathcal{T}(\gamma)(s) = \mathcal{T}(\gamma)(t)$ for all $\gamma = \beta[\, x_i \mapsto [t_i] \mid 1 \leq i \leq n\,]$ with $[t_i] \in U_{\mathcal{T}}$.

Choose $t_i = x_i$, then $[s] = \mathcal{T}(\gamma)(s) = \mathcal{T}(\gamma)(t) = [t]$, so $E \vdash s \approx t$ is derivable by definition of $\mathcal{T}$. □

Theorem 4.15 ("Birkhoff's Theorem") *Let* X *be a countably infinite set of variables, let* E *be a set of (universally quantified) equations. Then the following properties are equivalent for all* $s, t \in \mathrm{T}_\Sigma(X)$*:*

(i) $s \leftrightarrow_E^* t$.

(ii) $E \vdash s \approx t$ *is derivable.*

(iii) $s \approx_E t$, *i. e.,* $E \models \forall\vec{x}(s \approx t)$.

(iv) $\mathrm{T}_\Sigma(X)/E \models \forall\vec{x}(s \approx t)$.

Proof. (i)⇔(ii): Lemma 4.11.

(ii)⇒(iii): By induction on the size of the derivation for $E \vdash s \approx t$.

(iii)⇒(iv): Obvious, since $\mathcal{T} = \mathcal{T}_E(X)$ is an E-algebra.

(iv)⇒(ii): Lemma 4.14. □

Universal Algebra

$T_\Sigma(X)/E = T_\Sigma(X)/{\approx_E} = T_\Sigma(X)/{\leftrightarrow_E^*}$ is called the *free E-algebra* with generating set $X/{\approx_E} = \{\, [x] \mid x \in X \,\}$:

Every mapping $\varphi : X/{\approx_E} \to \mathcal{B}$ for some E-algebra $\mathcal{B}$ can be extended to a homomorphism $\hat{\varphi} : T_\Sigma(X)/E \to \mathcal{B}$.

$T_\Sigma(\emptyset)/E = T_\Sigma(\emptyset)/{\approx_E} = T_\Sigma(\emptyset)/{\leftrightarrow_E^*}$ is called the *initial E-algebra*.

${\approx_E} = \{\, (s,t) \mid E \models s \approx t \,\}$ is called the *equational theory* of E.

${\approx_E^I} = \{\, (s,t) \mid T_\Sigma(\emptyset)/E \models s \approx t \,\}$ is called the *inductive theory* of E.

Example:

Let $E = \{\forall x(x + 0 \approx x),\ \forall x \forall y(x + s(y) \approx s(x + y))\}$. Then $x + y \approx_E^I y + x$, but $x + y \not\approx_E y + x$.

Rewrite Relations

Corollary 4.16 *If E is convergent (i. e., terminating and confluent), then $s \approx_E t$ if and only if $s \leftrightarrow_E^* t$ if and only if $s{\downarrow_E} = t{\downarrow_E}$.*

Corollary 4.17 *If E is finite and convergent, then $\approx_E$ is decidable.*

Reminder:
If E is terminating, then it is confluent if and only if it is locally confluent.

Problems:

Show local confluence of E.

Show termination of E.

Transform E into an equivalent set of equations that is locally confluent and terminating.