Lemma 4.4 *If $\to$ is confluent, then every element has at most one normal form.*

Proof. Suppose that some element $a \in A$ has normal forms b and c, then $b \leftarrow^* a \to^* c$. If $\to$ is confluent, then $b \to^* d \leftarrow^* c$ for some $d \in A$. Since b and c are normal forms, both derivations must be empty, hence $b \to^0 d \leftarrow^0 c$, so b, c, and d must be identical. $\square$

Corollary 4.5 *If $\to$ is normalizing and confluent, then every element b has a unique normal form.*

Proposition 4.6 *If $\to$ is normalizing and confluent, then $b \leftrightarrow^* c$ if and only if $b{\downarrow} = c{\downarrow}$.*

Proof. Either using Thm. 4.3 or directly by induction on the length of the derivation of $b \leftrightarrow^* c$. $\square$

Well-Founded Orderings

Lemma 4.7 *If $\to$ is a terminating binary relation over A, then $\to^+$ is a well-founded partial ordering.*

Proof. Transitivity of $\to^+$ is obvious; irreflexivity and well-foundedness follow from termination of $\to$. $\square$

Lemma 4.8 *If $>$ is a well-founded partial ordering and $\to \subseteq >$, then $\to$ is terminating.*

Proving Confluence

Theorem 4.9 ("Newman's Lemma") *If a terminating relation $\to$ is locally confluent, then it is confluent.*

Proof. Let $\to$ be a terminating and locally confluent relation. Then $\to^+$ is a well-founded ordering. Define $P(a) \;\Leftrightarrow\; \big(\forall b, c : b \leftarrow^* a \to^* c \Rightarrow b \downarrow c\big)$.

We prove $P(a)$ for all $a \in A$ by well-founded induction over $\to^+$:

Case 1: $b \leftarrow^0 a \to^* c$: trivial.

Case 2: $b \leftarrow^* a \to^0 c$: trivial.

Case 3: $b \leftarrow^* x' \leftarrow a \to y' \to^* c$: use local confluence, then use the induction hypothesis. $\square$

Proving Termination: Monotone Mappings

Let $(A, >_A)$ and $(B, >_B)$ be partial orderings. A mapping $\varphi : A \to B$ is called *monotone*, if $a >_A a'$ implies $\varphi(a) >_B \varphi(a')$ for all $a, a' \in A$.

Lemma 4.10 *If $\varphi : A \to B$ is a monotone mapping from $(A, >_A)$ to $(B, >_B)$ and $(B, >_B)$ is well-founded, then $(A, >_A)$ is well-founded.*

4.3 Rewrite Systems

Let E be a set of equations.

The *rewrite relation* $\to_E \subseteq \mathrm{T}_\Sigma(X) \times \mathrm{T}_\Sigma(X)$ is defined by

$$
\begin{array}{lll}
s \to_E t & \text{iff} & \text{there exist } (l \approx r) \in E,\ p \in \mathrm{pos}(s), \\
 & & \text{and } \sigma : X \to \mathrm{T}_\Sigma(X), \\
 & & \text{such that } s/p = l\sigma \text{ and } t = s[r\sigma]_p.
\end{array}
$$

An instance of the lhs (left-hand side) of an equation is called a *redex* (reducible expression). *Contracting* a redex means replacing it with the corresponding instance of the rhs (right-hand side) of the rule.

An equation $l \approx r$ is also called a *rewrite rule*, if l is not a variable and $\mathrm{var}(l) \supseteq \mathrm{var}(r)$.

Notation: $l \to r$.

A set of rewrite rules is called a *term rewrite system (TRS)*.

We say that a set of equations E or a TRS R is terminating, if the rewrite relation $\to_E$ or $\to_R$ has this property.

(Analogously for other properties of abstract reduction systems).

Note: If E is terminating, then it is a TRS.

E-Algebras

Let E be a set of closed equations. A Σ-algebra $\mathcal{A}$ is called an *E-algebra*, if $\mathcal{A} \models \forall \vec{x}(s \approx t)$ for all $\forall \vec{x}(s \approx t) \in E$.

If $E \models \forall \vec{x}(s \approx t)$ (i. e., $\forall \vec{x}(s \approx t)$ is valid in all E-algebras), we write this also as $s \approx_E t$.

Goal:
Use the rewrite relation $\to_E$ to express the semantic consequence relation syntactically:

$s \approx_E t$ if and only if $s \leftrightarrow_E^* t$.