The solved form of $\Rightarrow_{PU}$ is different form the solved form obtained from $\Rightarrow_{SU}$. In order to obtain a unifier, the substitutions generated by the single equations have to be composed.

Lifting Lemma

Lemma 3.33 *Let C and D be variable-disjoint clauses. If*

$$\begin{array}{cc} D & C \\ \big\downarrow \sigma & \big\downarrow \rho \\ \dfrac{D\sigma \qquad C\rho}{C'} \end{array} \qquad \text{[propositional resolution]}$$

then there exists a substitution τ such that

$$\begin{array}{c} \dfrac{D \qquad C}{C''} \\ \big\downarrow \tau \\ C' = C''\tau \end{array} \qquad \text{[general resolution]}$$

An analogous lifting lemma holds for factorization.

Saturation of Sets of General Clauses

Corollary 3.34 *Let N be a set of general clauses saturated under Res, i. e., $Res(N) \subseteq N$. Then also $G_\Sigma(N)$ is saturated, that is,*

$$Res(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Proof. W.l.o.g. we may assume that clauses in N are pairwise variable-disjoint. (Otherwise make them disjoint, and this renaming process changes neither $Res(N)$ nor $G_\Sigma(N)$.)

Let $C' \in Res(G_\Sigma(N))$, meaning (i) there exist resolvable ground instances $D\sigma$ and $C\rho$ of N with resolvent C', or else (ii) C' is a factor of a ground instance $C\sigma$ of C.

Case (i): By the Lifting Lemma, D and C are resolvable with a resolvent C'' with $C''\tau = C'$, for a suitable substitution τ. As $C'' \in N$ by assumption, we obtain that $C' \in G_\Sigma(N)$.

Case (ii): Similar. □

Herbrand's Theorem

Lemma 3.35 *Let N be a set of Σ-clauses, let $\mathcal{A}$ be an interpretation. Then $\mathcal{A} \models N$ implies $\mathcal{A} \models G_\Sigma(N)$.*

Lemma 3.36 *Let N be a set of Σ-clauses, let $\mathcal{A}$ be a* Herbrand *interpretation. Then $\mathcal{A} \models G_\Sigma(N)$ implies $\mathcal{A} \models N$.*

Theorem 3.37 (Herbrand) *A set N of Σ-clauses is satisfiable if and only if it has a Herbrand model over Σ.*

Proof. The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let $N \not\models \bot$.

$$\begin{aligned} N \not\models \bot &\Rightarrow \bot \notin \mathit{Res}^*(N) \qquad \text{(resolution is sound)} \\ &\Rightarrow \bot \notin G_\Sigma(\mathit{Res}^*(N)) \\ &\Rightarrow I_{G_\Sigma(\mathit{Res}^*(N))} \models G_\Sigma(\mathit{Res}^*(N)) \qquad \text{(Thm. 3.24; Cor. 3.34)} \\ &\Rightarrow I_{G_\Sigma(\mathit{Res}^*(N))} \models \mathit{Res}^*(N) \qquad \text{(Lemma 3.36)} \\ &\Rightarrow I_{G_\Sigma(\mathit{Res}^*(N))} \models N \qquad (N \subseteq \mathit{Res}^*(N)) \quad \Box \end{aligned}$$

The Theorem of Löwenheim-Skolem

Theorem 3.38 (Löwenheim–Skolem) *Let Σ be a countable signature and let S be a set of closed Σ-formulas. Then S is satisfiable iff S has a model over a countable universe.*

Proof. If both X and Σ are countable, then S can be at most countably infinite. Now generate, maintaining satisfiability, a set N of clauses from S. This extends Σ by at most countably many new Skolem functions to Σ'. As Σ' is countable, so is $T_{\Sigma'}$, the universe of Herbrand-interpretations over Σ'. Now apply Theorem 3.37. □

Refutational Completeness of General Resolution

Theorem 3.39 *Let N be a set of general clauses where $\mathit{Res}(N) \subseteq N$. Then*

$$N \models \bot \Leftrightarrow \bot \in N.$$

Proof. Let $\mathit{Res}(N) \subseteq N$. By Corollary 3.34: $\mathit{Res}(G_\Sigma(N)) \subseteq G_\Sigma(N)$

$$\begin{aligned} N \models \bot &\Leftrightarrow G_\Sigma(N) \models \bot \qquad \text{(Lemma 3.35/3.36; Theorem 3.37)} \\ &\Leftrightarrow \bot \in G_\Sigma(N) \qquad \text{(propositional resolution sound and complete)} \\ &\Leftrightarrow \bot \in N \quad \Box \end{aligned}$$

Compactness of Predicate Logic

Theorem 3.40 (Compactness Theorem for First-Order Logic) *Let Φ be a set of first-order formulas. Φ is unsatisfiable $\Leftrightarrow$ some finite subset $\Psi \subseteq \Phi$ is unsatisfiable.*

Proof. The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let Φ be unsatisfiable and let N be the set of clauses obtained by Skolemization and CNF transformation of the formulas in Φ. Clearly $Res^*(N)$ is unsatisfiable. By Theorem 3.39, $\bot \in Res^*(N)$, and therefore $\bot \in Res^n(N)$ for some $n \in \mathbb{N}$. Consequently, $\bot$ has a finite resolution proof B of depth $\leq n$. Choose Ψ as the subset of formulas in Φ such that the corresponding clauses contain the assumptions (leaves) of B. $\square$

3.13 Ordered Resolution with Selection

Motivation: Search space for *Res very* large.

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem 3.24) one only needs to resolve and factor maximal atoms
$\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
$\Rightarrow$ *order restrictions*
2. In the proof, it does not really matter with which negative literal an inference is performed
$\Rightarrow$ choose a negative literal don't-care-nondeterministically
$\Rightarrow$ *selection*

Selection Functions

A *selection function* is a mapping

$$S : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \vee \neg A \vee B$$

$$\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$$

Resolution Calculus $Res_S^{\succ}$

In the completeness proof, we talk about (strictly) maximal literals of *ground* clauses.

In the non-ground calculus, we have to consider those literals that correspond to (strictly) maximal literals of ground instances:

Let $\succ$ be a total and well-founded ordering on ground atoms. A literal L is called *[strictly] maximal* in a clause C if and only if there exists a ground substitution σ such that for no other L' in C: $L\sigma \prec L'\sigma$ $[L\sigma \preceq L'\sigma]$.

Let $\succ$ be an atom ordering and S a selection function.

$$\frac{D \lor B \qquad C \lor \neg A}{(D \lor C)\sigma} \qquad [\textit{ordered resolution with selection}]$$

if $\sigma = \text{mgu}(A, B)$ and

(i) $B\sigma$ strictly maximal w. r. t. $D\sigma$;

(ii) nothing is selected in D by S;

(iii) either $\neg A$ is selected, or else nothing is selected in $C \lor \neg A$ and $\neg A\sigma$ is maximal in $C\sigma$.

$$\frac{C \lor A \lor B}{(C \lor A)\sigma} \qquad [\textit{ordered factoring}]$$

if $\sigma = \text{mgu}(A, B)$ and $A\sigma$ is maximal in $C\sigma$ and nothing is selected in C.

Special Case: Propositional Logic

For ground clauses the resolution inference simplifies to

$$\frac{D \lor A \qquad C \lor \neg A}{D \lor C}$$

if

(i) $A \succ D$;

(ii) nothing is selected in D by. S;

(iii) $\neg A$ is selected in $C \lor \neg A$, or else nothing is selected in $C \lor \neg A$ and $\neg A \succeq \max(C)$.

Note: For positive literals, $A \succ D$ is the same as $A \succ \max(D)$.

Search Spaces Become Smaller

1	$A \vee B$		we assume $A \succ B$ and S as indicated by $\boxed{X}$. The maximal literal in a clause is depicted in red.
2	$A \vee \boxed{\neg B}$		
3	$\neg A \vee B$		
4	$\neg A \vee \boxed{\neg B}$		
5	$B \vee B$	Res 1, 3	
6	B	Fact 5	
7	$\neg A$	Res 6, 4	
8	A	Res 6, 2	
9	$\bot$	Res 8, 7	

With this ordering and selection function the refutation proceeds strictly deterministically in this example. Generally, proof search will still be non-deterministic but the search space will be much smaller than with unrestricted resolution.

Avoiding Rotation Redundancy

From

$$\dfrac{\dfrac{C_1 \vee A \quad C_2 \vee \neg A \vee B}{C_1 \vee C_2 \vee B} \qquad C_3 \vee \neg B}{C_1 \vee C_2 \vee C_3}$$

we can obtain by *rotation*

$$\dfrac{C_1 \vee A \qquad \dfrac{C_2 \vee \neg A \vee B \quad C_3 \vee \neg B}{C_2 \vee \neg A \vee C_3}}{C_1 \vee C_2 \vee C_3}$$

another proof of the same clause. In large proofs many rotations are possible. However, if $A \succ B$, then the second proof does not fulfill the orderings restrictions.

Conclusion: In the presence of orderings restrictions (however one chooses $\succ$) no rotations are possible. In other words, orderings identify exactly one representant in any class of of rotation-equivalent proofs.

Lifting Lemma for $Res_S^{\succ}$

Lemma 3.41 *Let D and C be variable-disjoint clauses. If*

$$\begin{array}{cc} D & C \\ \downarrow \sigma & \downarrow \rho \\ \multicolumn{2}{c}{\dfrac{D\sigma \qquad C\rho}{C'}} \end{array} \qquad \text{[propositional inference in } Res_S^{\succ}\text{]}$$