Simplification Orderings

The proper subterm ordering $\rhd$ is defined by $s \rhd t$ if and only if $s/p = t$ for some position $p \neq \varepsilon$ of s.

Simplification Orderings

A rewrite ordering $\succ$ over $\mathrm{T}_\Sigma(X)$ is called simplification ordering, if it has the subterm property:
$s \rhd t$ implies $s \succ t$ for all $s, t \in \mathrm{T}_\Sigma(X)$.

Example:

Let R_{emb} be the rewrite system
$R_{\text{emb}} = \{\, f(x_1, \dots, x_n) \to x_i \mid f/n \in \Omega,\ n \geq 1,\ 1 \leq i \leq n \,\}$.

Define $\rhd_{\text{emb}} = \to^{+}_{R_{\text{emb}}}$ and $\unrhd_{\text{emb}} = \to^{*}_{R_{\text{emb}}}$
("homeomorphic embedding relation").

$\rhd_{\text{emb}}$ is a simplification ordering.

Simplification Orderings

Lemma 3.31:

If $\succ$ is a simplification ordering, then $s \rhd_{\text{emb}} t$ implies $s \succ t$ and $s \unrhd_{\text{emb}} t$ implies $s \succeq t$.

Proof:

Since $\succ$ is transitive and $\succeq$ is transitive and reflexive, it suffices to show that $s \rightarrow_{R_{\text{emb}}} t$ implies $s \succ t$.

By definition, $s \rightarrow_{R_{\text{emb}}} t$ if and only if $s = s[l\sigma]$ and $t = s[r\sigma]$ for some rule $l \rightarrow r \in R_{\text{emb}}$.

Obviously, $l \rhd r$ for all rules in R_{emb}, hence $l \succ r$.

Since $\succ$ is a rewrite relation, $s = s[l\sigma] \succ s[r\sigma] = t$.

Simplification Orderings

Goal:

Show that every simplification ordering is well-founded (and therefore a reduction ordering).

Note: This works only for finite signatures!

To fix this for infinite signatures, the definition of simplification orderings and the definition of embedding have to be modified.

Kruskal's Theorem

A (usually not strict) partial ordering $\succeq$ on a set A is called well-partial-ordering (wpo), if for every infinite sequence $a_1, a_2, a_3, \ldots$ there are indices $i < j$ such that $a_i \preceq a_j$.

Terminology:

An infinite sequence $a_1, a_2, a_3, \ldots$ is called good, if there exist $i < j$ such that $a_i \preceq a_j$; otherwise it is called bad.

Therefore: $\succeq$ is a wpo iff every infinite sequence is good.

Kruskal's Theorem

Lemma 3.32:

If $\succeq$ is a wpo, then every infinite sequence $a_1, a_2, a_3, \ldots$ has an infinite ascending subsequence $a_{i_1} \preceq a_{i_2} \preceq a_{i_3} \preceq \ldots$, where $i_1 < i_2 < i_3 < \ldots$.

Proof:

Let $a_1, a_2, a_3, \ldots$ be an infinite sequence. We call an index $m \geq 1$ terminal, if there is no $n > m$ such that $a_m \preceq a_n$.
There are only finitely many terminal indices $m_1, m_2, m_3, \ldots$; otherwise the sequence $a_{m_1}, a_{m_2}, a_{m_3}, \ldots$ would be bad.
Choose $p > 1$ such that all $m \geq p$ are not terminal; define $i_1 = p$; define recursively i_{j+1} such that $i_{j+1} > i_j$ and $a_{i_{j+1}} \succeq a_{i_j}$.

Kruskal's Theorem

Lemma 3.33:

If $\succeq_1, \ldots, \succeq_n$ are wpo's on $A_1, \ldots, A_n$, then $\succeq$ defined by

$$(a_1, \ldots, a_n) \succeq (a'_1, \ldots, a'_n) \text{ iff } a_i \succeq_i a'_i \text{ for all } i$$

is a wpo on $A_1 \times \cdots \times A_n$.

Proof:

The case $n = 1$ is trivial.
Otherwise let $(a_1^{(1)}, \ldots, a_n^{(1)}), (a_1^{(2)}, \ldots, a_n^{(2)}), \ldots$ be an infinite sequence. By the previous lemma, there are infinitely many indices $i_1 < i_2 < i_3 < \ldots$ such that $a_n^{(i_1)} \preceq a_n^{(i_2)} \preceq a_n^{(i_3)} \preceq \ldots$.
By induction on n, there are $k < l$ such that $a_1^{(i_k)} \preceq a_1^{(i_l)} \wedge \cdots \wedge a_{n-1}^{(i_k)} \preceq a_{n-1}^{(i_l)}$. Therefore $(a_1^{(i_k)}, \ldots, a_n^{(i_k)}) \preceq (a_1^{(i_l)}, \ldots, a_n^{(i_l)})$.

Kruskal's Theorem

Theorem 3.34 ("Kruskal's Theorem"):

Let Σ be a finite signature, let X be a finite set of variables.

Then $\trianglerighteq_{\text{emb}}$ is a wpo on $\mathsf{T}_\Sigma(X)$.

Proof:

Baader and Nipkow, page 114/115.

Simplification Orderings

Theorem 3.35 (Dershowitz):
If Σ is a finite signature, then every simplification ordering $\succ$ on $\mathrm{T}_\Sigma(X)$ is well-founded (and therefore a reduction ordering).

Proof:
Suppose that $t_1 \succ t_2 \succ t_3 \succ \ldots$ is an infinite decreasing chain.

First assume that there is an $x \in \mathrm{var}(t_{i+1}) \setminus \mathrm{var}(t_i)$.
Let $\sigma = [t_i/x]$, then $t_{i+1}\sigma \trianglerighteq x\sigma = t_i$ and therefore $t_i = t_i\sigma \succ t_{i+1}\sigma \succeq t_i$, contradicting reflexivity.

Consequently, $\mathrm{var}(t_i) \supseteq \mathrm{var}(t_{i+1})$ and $t_i \in \mathrm{T}_\Sigma(V)$ for all i, where V is the finite set $\mathrm{var}(t_1)$. By Kruskal's Theorem, there are $i < j$ with $t_i \trianglelefteq_{\mathrm{emb}} t_j$. Hence $t_i \preceq t_j$, contradicting $t_i \succ t_j$.

Simplification Orderings

There are reduction orderings that are not simplification orderings and terminating TRSs that are not contained in any simplification ordering.

Example:

Let $R = \{f(f(x)) \to f(g(f(x)))\}$.

R terminates and $\to_R^+$ is therefore a reduction ordering.

Assume that $\to_R$ were contained in a simplification ordering $\succ$. Then $f(f(x)) \to_R f(g(f(x)))$ implies $f(f(x)) \succ f(g(f(x)))$, and $f(g(f(x))) \trianglerighteq_{\text{emb}} f(f(x))$ implies $f(g(f(x))) \succeq f(f(x))$, hence $f(f(x)) \succ f(f(x))$.

Recursive Path Orderings

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering ("precedence") on Ω.

The lexicographic path ordering $\succ_{\mathrm{lpo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{\mathrm{lpo}} t$ iff

(1) $t \in \mathrm{var}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

- (a) $s_i \succeq_{\mathrm{lpo}} t$ for some i, or
- (b) $f \succ g$ and $s \succ_{\mathrm{lpo}} t_j$ for all j, or
- (c) $f = g$, $s \succ_{\mathrm{lpo}} t_j$ for all j, and $(s_1, \ldots, s_m)\ (\succ_{\mathrm{lpo}})_{\mathrm{lex}}\ (t_1, \ldots, t_n)$.

Recursive Path Orderings

Lemma 3.36:

$s \succ_{\mathsf{lpo}} t$ implies $\mathrm{var}(s) \supseteq \mathrm{var}(t)$.

Proof:

By induction on $|s| + |t|$ and case analysis.

Recursive Path Orderings

Theorem 3.37:
$\succ_{\mathsf{lpo}}$ is a simplification ordering on $\mathsf{T}_\Sigma(X)$.

Proof:
Show transitivity, subterm property, stability under substitutions, compatibility with Σ-operations, and irreflexivity, usually by induction on the sum of the term sizes and case analysis.
Details: Baader and Nipkow, page 119/120.

Recursive Path Orderings

Theorem 3.38:

If the precedence $\succ$ is total, then the lexicographic path ordering $\succ_{\mathsf{lpo}}$ is total on ground terms, i. e., for all $s, t \in \mathsf{T}_\Sigma(\emptyset)$:
$s \succ_{\mathsf{lpo}} t \vee t \succ_{\mathsf{lpo}} s \vee s = t$.

Proof:

By induction on $|s| + |t|$ and case analysis.

Recursive Path Orderings

Recapitulation:

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering ("precedence") on Ω. The lexicographic path ordering $\succ_{\mathsf{lpo}}$ on $\mathsf{T}_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{\mathsf{lpo}} t$ iff

(1) $t \in \mathrm{var}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

- (a) $s_i \succeq_{\mathsf{lpo}} t$ for some i, or
- (b) $f \succ g$ and $s \succ_{\mathsf{lpo}} t_j$ for all j, or
- (c) $f = g$, $s \succ_{\mathsf{lpo}} t_j$ for all j, and $(s_1, \ldots, s_m)\ (\succ_{\mathsf{lpo}})_{\mathsf{lex}}\ (t_1, \ldots, t_n)$.

Recursive Path Orderings

There are several possibilities to compare subterms in (2)(c):

compare list of subterms lexicographically left-to-right
("lexicographic path ordering (lpo)", Kamin and Lévy)

compare list of subterms lexicographically right-to-left
(or according to some permutation π)

compare multiset of subterms using the multiset extension
("multiset path ordering (mpo)", Dershowitz)

to each function symbol f/n associate a
status $\in \{mul\} \cup \{ lex_\pi \mid \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \}$
and compare according to that status
("recursive path ordering (rpo) with status")

The Knuth-Bendix Ordering

Let $\Sigma = (\Omega, \Pi)$ be a finite signature,
let $\succ$ be a strict partial ordering ("precedence") on Ω,
let $w : \Omega \cup X \to \mathbb{R}_0^+$ be a weight function,
such that the following admissibility conditions are satisfied:

$w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in X$;
$w(c) \geq w_0$ for all constants $c/0 \in \Omega$.

If $w(f) = 0$ for some $f/1 \in \Omega$, then $f \succeq g$ for all $g \in \Omega$.

w can be extended to terms as follows:

$$w(t) = \sum_{x \in \text{var}(t)} w(x) \cdot \#(x, t) + \sum_{f \in \Omega} w(f) \cdot \#(f, t).$$

The Knuth-Bendix Ordering

The Knuth-Bendix ordering $\succ_{\mathrm{kbo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $\succ$ and w is defined by: $s \succ_{\mathrm{kbo}} t$ iff

(1) $\#(x, s) \geq \#(x, t)$ for all variables x and $w(s) > w(t)$, or

(2) $\#(x, s) \geq \#(x, t)$ for all variables x, $w(s) = w(t)$, and

   (a) $t = x$, $s = f^n(x)$ for some $n \geq 1$, or

   (b) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f \succ g$, or

   (c) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and $(s_1, \ldots, s_m) \, (\succ_{\mathrm{kbo}})_{\mathrm{lex}} \, (t_1, \ldots, t_m)$.

The Knuth-Bendix Ordering

Theorem 3.39:
The Knuth-Bendix ordering induced by $\succ$ and w is a simplification ordering on $\mathrm{T}_\Sigma(X)$.

Proof:
Baader and Nipkow, pages 125–129.

3.6 Knuth-Bendix Completion

Completion:

Goal: Given a set E of equations, transform E into an equivalent convergent set R of rewrite rules.

How to ensure termination?

Fix a reduction ordering $\succ$ and construct R in such a way that $\rightarrow_R \subseteq \succ$ (i. e., $l \succ r$ for every $l \rightarrow r \in R$).

How to ensure confluence?

Check that all critical pairs are joinable.

Knuth-Bendix Completion: Inference Rules

The completion procedure is presented as a set of inference rules working on a set of equations E and a set of rules R:
$E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$

At the beginning, $E = E_0$ is the input set and $R = R_0$ is empty. At the end, E should be empty; then R is the result.

For each step $E, R \vdash E', R'$, the equational theories of $E \cup R$ and $E' \cup R'$ agree: $\approx_{E \cup R} = \approx_{E' \cup R'}$.

Knuth-Bendix Completion: Inference Rules

Notations:

The formula $s \dot{\approx} t$ denotes either $s \approx t$ or $t \approx s$.

$\mathrm{CP}(R)$ denotes the set of all critical pairs between rules in R.

Knuth-Bendix Completion: Inference Rules

Orient:

$$\frac{E \cup \{s \mathrel{\dot{\approx}} t\}, \quad R}{E, \quad R \cup \{s \rightarrow t\}} \qquad \text{if } s \succ t$$

Note: There are equations $s \approx t$ that cannot be oriented, i. e., neither $s \succ t$ nor $t \succ s$.

Knuth-Bendix Completion: Inference Rules

Trivial equations cannot be oriented – but we don't need them anyway:

Delete:

$$\frac{E \cup \{s \approx s\}, \quad R}{E, \quad R}$$

Knuth-Bendix Completion: Inference Rules

Critical pairs between rules in R are turned into additional equations:

Deduce:

$$\frac{E, \quad R}{E \cup \{s \approx t\}, \quad R} \qquad \text{if } \langle s, t \rangle \in \mathrm{CP}(R).$$

Note: If $\langle s, t \rangle \in \mathrm{CP}(R)$ then $s \leftarrow_R u \rightarrow_R t$ and hence $R \models s \approx t$.

Knuth-Bendix Completion: Inference Rules

The following inference rules are not absolutely necessary, but very useful (e.g., to get rid of joinable critical pairs and to deal with equations that cannot be oriented):

Simplify-Eq:

$$\frac{E \cup \{s \mathrel{\dot{\approx}} t\}, \quad R}{E \cup \{u \approx t\}, \quad R} \qquad \text{if } s \rightarrow_R u.$$

Knuth-Bendix Completion: Inference Rules

Simplification of the right-hand side of a rule is unproblematic.

R-Simplify-Rule:

$$\frac{E, \quad R \cup \{s \to t\}}{E, \quad R \cup \{s \to u\}} \qquad \text{if } t \to_R u.$$

Simplification of the left-hand side may influence orientability and orientation. Therefore, it yields an *equation*:

L-Simplify-Rule:

$$\frac{E, \quad R \cup \{s \to t\}}{E \cup \{u \approx t\}, \quad R}$$

if $s \to_R u$ using a rule $l \to r \in R$ such that $s \sqsupset l$ (see next slide).

Knuth-Bendix Completion: Inference Rules

For technical reasons, the lhs of $s \to t$ may only be simplified using a rule $l \to r$, if $l \to r$ *cannot* be simplified using $s \to t$, that is, if $s \sqsupset l$, where the encompassment quasi-ordering $\sqsupset\!\!\sim$ is defined by

$$s \mathrel{\underset{\sim}{\sqsupset}} l \quad \text{if} \quad s/p = l\sigma \text{ for some } p \text{ and } \sigma$$

and $\sqsupset \; = \; \underset{\sim}{\sqsupset} \setminus \underset{\sim}{\sqsubset}$ is the strict part of $\underset{\sim}{\sqsupset}$.

Lemma 3.40:

$\sqsupset$ is a well-founded strict partial ordering.

Knuth-Bendix Completion: Inference Rules

Lemma 3.41:
If $E, R \vdash E', R'$, then $\approx_{E \cup R} = \approx_{E' \cup R'}$.

Lemma 3.42:
If $E, R \vdash E', R'$ and $\rightarrow_R \subseteq \succ$, then $\rightarrow_{R'} \subseteq \succ$.

Knuth-Bendix Completion: Correctness Proof

If we run the completion procedure on a set E of equations, different things can happen:

(1) We reach a state where no more inference rules are applicable and E is not empty.
$\Rightarrow$ Failure (try again with another ordering?)

(2) We reach a state where E is empty and all critical pairs between the rules in the current R have been checked.

(3) The procedure runs forever.

In order to treat these cases simultaneously, we need some definitions.

Knuth-Bendix Completion: Correctness Proof

A (finite or infinite sequence) $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$ with $R_0 = \emptyset$ is called a run of the completion procedure with input E_0 and $\succ$.

For a run, $E_\infty = \bigcup_{i \geq 0} E_i$ and $R_\infty = \bigcup_{i \geq 0} R_i$.

The sets of persistent equations or rules of the run are $E_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} E_j$ and $R_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} R_j$.

Note: If the run is finite and ends with E_n, R_n, then $E_* = E_n$ and $R_* = R_n$.

Knuth-Bendix Completion: Correctness Proof

A run is called fair, if $CP(R_*) \subseteq E_\infty$
(i. e., if every critical pair between persisting rules is computed at some step of the derivation).

Goal:

Show: If a run is fair and E_* is empty,
then R_* is convergent and equivalent to E_0.

In particular: If a run is fair and E_* is empty,
then $\approx_{E_0} = \approx_{E_\infty \cup R_\infty} = \leftrightarrow_{E_\infty \cup R_\infty} = \downarrow_{R_*}$.

Knuth-Bendix Completion: Correctness Proof

General assumptions from now on:

$E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$ is a fair run.

R_0 and E_* are empty.

Knuth-Bendix Completion: Correctness Proof

A proof of $s \approx t$ in $E_\infty \cup R_\infty$ is a finite sequence $(s_0, \ldots, s_n)$ such that $s = s_0$, $t = s_n$, and for all $i \in \{1, \ldots, n\}$:

(1) $s_{i-1} \leftrightarrow_{E_\infty} s_i$, or

(2) $s_{i-1} \rightarrow_{R_\infty} s_i$, or

(3) $s_{i-1} \leftarrow_{R_\infty} s_i$.

The pairs (s_{i-1}, s_i) are called proof steps.

A proof is called a rewrite proof in R_*,
if there is a $k \in \{0, \ldots, n\}$ such that $s_{i-1} \rightarrow_{R_*} s_i$ for $1 \leq i \leq k$ and $s_{i-1} \leftarrow_{R_*} s_i$ for $k + 1 \leq i \leq n$

Knuth-Bendix Completion: Correctness Proof

Idea (Bachmair, Dershowitz, Hsiang):

Define a well-founded ordering on proofs, such that for every proof that is not a rewrite proof in R_* there is an equivalent smaller proof.

Consequence: For every proof there is an equivalent rewrite proof in R_*.

Knuth-Bendix Completion: Correctness Proof

We associate a cost $c(s_{i-1}, s_i)$ with every proof step as follows:

(1) If $s_{i-1} \leftrightarrow_{E_\infty} s_i$, then $c(s_{i-1}, s_i) = (\{s_{i-1}, s_i\}, -, -)$, where the first component is a multiset of terms and $-$ denotes an arbitrary (irrelevant) term.

(2) If $s_{i-1} \rightarrow_{R_\infty} s_i$ using $l \rightarrow r$, then $c(s_{i-1}, s_i) = (\{s_{i-1}\}, l, s_i)$.

(3) If $s_{i-1} \leftarrow_{R_\infty} s_i$ using $l \rightarrow r$, then $c(s_{i-1}, s_i) = (\{s_i\}, l, s_{i-1})$.

Proof steps are compared using the lexicographic combination of the multiset extension of reduction ordering $\succ$, the encompassment ordering $\sqsupset$, and the reduction ordering $\succ$.

Knuth-Bendix Completion: Correctness Proof

The cost $c(P)$ of a proof P is the multiset of the costs of its proof steps.

The proof ordering $\succ_C$ compares the costs of proofs using the multiset extension of the proof step ordering.

Lemma 3.43:
$\succ_C$ is a well-founded ordering.

Knuth-Bendix Completion: Correctness Proof

Lemma 3.44:
Let P be a proof in $E_\infty \cup R_\infty$. If P is not a rewrite proof in R_*, then there exists an equivalent proof P' in $E_\infty \cup R_\infty$ such that $P \succ_C P'$.

Proof:
If P is not a rewrite proof in R_*, then it contains

(a) a proof step that is in E_∞, or
(b) a proof step that is in $R_\infty \setminus R_*$, or
(c) a subproof $s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1}$ (peak).

We show that in all three cases the proof step or subproof can be replaced by a smaller subproof:

Knuth-Bendix Completion: Correctness Proof

Case (a): A proof step using an equation $s \mathrel{\dot{\approx}} t$ is in E_∞.
This equation must be deleted during the run.

If $s \mathrel{\dot{\approx}} t$ is deleted using *Orient*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_i \dots \qquad \Longrightarrow \qquad \dots s_{i-1} \rightarrow_{R_\infty} s_i \dots$$

If $s \mathrel{\dot{\approx}} t$ is deleted using *Delete*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_{i-1} \dots \qquad \Longrightarrow \qquad \dots s_{i-1} \dots$$

If $s \mathrel{\dot{\approx}} t$ is deleted using *Simplify-Eq*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_i \dots \qquad \Longrightarrow \qquad \dots s_{i-1} \rightarrow_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \dots$$

Knuth-Bendix Completion: Correctness Proof

Case (b): A proof step using a rule $s \to t$ is in $R_\infty \setminus R_*$.
This rule must be deleted during the run.

If $s \to t$ is deleted using *R-Simplify-Rule*:

$$\ldots s_{i-1} \to_{R_\infty} s_i \ldots \quad\Longrightarrow\quad \ldots s_{i-1} \to_{R_\infty} s' \leftarrow_{R_\infty} s_i \ldots$$

If $s \to t$ is deleted using *L-Simplify-Rule*:

$$\ldots s_{i-1} \to_{R_\infty} s_i \ldots \quad\Longrightarrow\quad \ldots s_{i-1} \to_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \ldots$$

Knuth-Bendix Completion: Correctness Proof

Case (c): A subproof has the form $s_{i-1} \longleftarrow_{R_*} s_i \longrightarrow_{R_*} s_{i+1}$.

If there is no overlap or a non-critical overlap:

$$\ldots s_{i-1} \longleftarrow_{R_*} s_i \longrightarrow_{R_*} s_{i+1} \ldots \implies \ldots s_{i-1} \longrightarrow^*_{R_*} s' \longleftarrow^*_{R_*} s_{i+1} \ldots$$

If there is a critical pair that has been added using *Deduce*:

$$\ldots s_{i-1} \longleftarrow_{R_*} s_i \longrightarrow_{R_*} s_{i+1} \ldots \implies \qquad \ldots s_{i-1} \longleftrightarrow_{E_\infty} s_i \ldots$$

In all cases, checking that the replacement subproof is smaller than the replaced subproof is routine.

Knuth-Bendix Completion: Correctness Proof

Theorem 3.45:
Let $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$ be a fair run and let R_0 and E_* be empty. Then

(1) every proof in $E_\infty \cup R_\infty$ is equivalent to a rewrite proof in R_*,

(2) R_* is equivalent to E_0, and

(3) R_* is convergent.

Knuth-Bendix Completion: Correctness Proof

Proof:

(1) By well-founded induction on $\succ_C$ using the previous lemma.

(2) Clearly $\approx_{E_\infty \cup R_\infty} = \approx_{E_0}$.
Since $R_* \subseteq R_\infty$, we get $\approx_{R_*} \subseteq \approx_{E_\infty \cup R_\infty}$.
On the other hand, by (1), $\approx_{E_\infty \cup R_\infty} \subseteq \approx_{R_*}$.

(3) Since $\rightarrow_{R_*} \subseteq \succ$, R_* is terminating.
By (1), R_* is confluent.

Knuth-Bendix Completion: Outlook

Classical completion:

Fails, if an equation can neither be oriented nor deleted.

Unfailing Completion:

Use an ordering $\succ$ that is total on ground terms.

If an equation cannot be oriented, use it in both directions for rewriting (except if that would yield a larger term).
In other words, consider the relation $\leftrightarrow_E \cap \not\prec$.

Special case of superposition (see next chapter).