Problem 1 *(Semantics)* (14 points)

Show that the following inference rule is sound:

$$\frac{D \ \vee \ f(s) \approx s' \qquad C \ \vee \ f(t) \approx t'}{D \ \vee \ C \ \vee \ s \not\approx t \ \vee \ s' \approx t'}$$

where the premises and conclusions are (implicitly universally quantified) equational clauses and the two premises have no common variables.

Problem 2 *(Ordered resolution with selection)* (10 points)

Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{a/0,\, b/0,\, f/1,\, g/1\}$ and $\Pi = \{p/1,\, q/1,\, r/2\}$. Let N be the following set of clauses:

$$\neg p(g(a)) \qquad (1)$$
$$q(f(x)) \qquad (2)$$
$$r(x, x) \qquad (3)$$
$$\neg q(g(x)) \ \vee \ p(g(x)) \qquad (4)$$
$$\neg r(a, z) \ \vee \ q(a) \ \vee \ p(f(z)) \qquad (5)$$
$$\neg r(f(y), y) \ \vee \ \neg q(y) \ \vee \ \neg p(f(f(y))) \qquad (6)$$

Suppose that an ordering $\succ$ on ground atoms is defined in such a way that $p(\dots) \succ q(\dots) \succ r(\dots)$. Which literals in which clauses must be selected by a selection function S such that N is saturated under $Res^{\succ}_{S}$?

Problem 3 *(Tableaux)* (16 points)

Use semantic tableaux to show that the following set of formulas is unsatisfiable:

$$\forall x \left(\exists y \; p(x, y) \rightarrow \exists z \; p(f(x), z) \right)$$
$$p(a, a)$$
$$\neg \, \exists x \; p(f(f(a)), x)$$

(Note: Quantifiers extend over the shortest following subformula, or in other words, $\exists y \; F \rightarrow \exists z \; G$ means $(\exists y \; F) \rightarrow (\exists z \; G)$.)

Problem 4 *(E-algebras)* (8 + 8 = 16 points)

Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{a/0,\ b/0,\ f/1\}$; let E be the set of (implicitly universally quantified) equations $\{\, f(f(x)) \approx a \,\}$.

Part (a)

Give one possible derivation for the statement $E \vdash f(a) \approx a$.

Part (b)

Is the universe of the initial E-algebra $\mathrm{T}_\Sigma(\emptyset)/E$ finite or infinite? If it is finite, how many elements does it have?

Problem 5 *(Reduction orderings)* (10 points)

Let Σ be an arbitrary first-order signature. Define the ordering $\succ$ on $\mathrm{T}_\Sigma(X)$ by

$$s \succ s' \ \text{ if and only if } \ |s| > |s'|$$

where $|t|$ is the size of t, that is, the cardinality of $\mathrm{pos}(t)$. Is $\succ$ a reduction ordering? Give a proof or a counterexample.

Problem 6 *(Knuth-Bendix completion)* (14 points)

Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{a/0,\ b/0,\ f/1,\ g/2,\ h/1\}$, let $\succ$ be the LPO with precedence $h > g > f > b > a$, and let E be the following set of equations:

$$f(x) \approx g(h(x), x) \qquad (1)$$
$$g(a, b) \approx b \qquad (2)$$
$$h(b) \approx a \qquad (3)$$

Use Knuth-Bendix completion to transform E into an equivalent convergent set R of rewrite rules such that $\rightarrow_R \subseteq \succ$.